Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 258 (2015) 1791–1800

Journal of Differential Equations

www.elsevier.com/locate/jde

From isochronous potentials to isochronous systems

Andrea Sfecci

Università Politecnica delle Marche, Dipartimento di Ingegneria Industriale e Scienze Matematiche, Via Brecce Bianche 12, I-60131 Ancona, Italy

Received 25 February 2014

Available online 1 December 2014

Abstract

There is a wide literature involving the study of isochronous equations of the type

$$\ddot{x}(t) + V'\big(x(t)\big) = 0,$$

where V is a C^2-function. In this paper we show how the kinetic energy $T(y) = \frac{1}{2}y^2$ can be modified still preserving the isochronicity property of the corresponding system. More generally we provide estimates for the periods, and show an application to the Steen's equation and other systems related to the anharmonic potential $V(x) = ax^2 + bx^{-2}$.

MSC: 34C25

Keywords: Isochronous potential; Periodic solution; Planar system; Steen's equation

1. Introduction

The study of *isochronous centers* for differential systems of the type

$$\begin{cases} \dot{x} = \mathcal{P}(x, y) \\ \dot{y} = \mathcal{Q}(x, y), \end{cases} \tag{1}$$

E-mail address: sfecci@dipmat.univpm.it.

http://dx.doi.org/10.1016/j.jde.2014.11.013

presents a wide literature (see e.g. [8,11] and the references therein). In particular, in this paper, we consider Hamiltonian systems of the type

$$\begin{cases} -\dot{y} = V'(x) \\ \dot{x} = T'(y), \end{cases} \tag{2}$$

where T and V are C^2-functions, with associated Hamiltonian function

$$E(x, y) = T(y) + V(x). \tag{3}$$

The simple case $T(y) = \frac{1}{2}y^2$ is related to the scalar second order differential equation

$$\ddot{x}(t) + V'\big(x(t)\big) = 0, \tag{4}$$

where V is a C^2-function with a strict local minimum point, where V is zero. This equation has been studied widely in literature, see e.g. [4,17,19,21,25,26]. In particular, all the periodic solutions have the period

$$\tau_0 = \sqrt{2} \int_{x_1}^{x_2} \frac{dx}{\sqrt{E - V(x)}}, \tag{5}$$

where E is the energy of the solution and $x_1 < x_2$ are the extremals of the orbit. The study of functions V providing a constant period function τ leads to the following definition.

Definition 1.1 *(Isochronous potential).* A real valued C^2-function $A = A(u)$, defined on an open interval I, is said to be an isochronous potential if the equation

$$\ddot{u}(t) + A'\big(u(t)\big) = 0 \tag{6}$$

has a unique equilibrium point and all the other solutions such that $u(t) \in I$ for every $t \in \mathbb{R}$, are periodic of the same period.

In 1961, Urabe [25] proved that the harmonic potential $V(x) = \frac{1}{2}k(x - x_0)^2$ (with $k \neq 0$) is the unique isochronous analytic potential, such that V' is odd. Later on, it has been proved that it is also the unique isochronous potential among polynomials [2,9]. A different approach, introducing *strict involutions*, has been introduced in [11]. See also [1,10,14–18,23,26] for other results. There are several well-known isochronous potentials $A(u)$, for instance, besides the harmonic one au^2, we have the anharmonic one $au^2 + bu^{-2}$, with $u > 0$, where the constants are assumed to be positive. It has been recently proved in [7], that all the rational isochronous potentials have one of these forms up to constants and shifts. An example of non-rational isochronous potential has been given by Urabe [25]: $A(u) = 1 + u - \sqrt{1 + 2u}$ for $u > -1/2$. Other examples can be given where the isochronicity property only holds locally, i.e. in a neighborhood of an equilibrium point. The results established below for the global case could be extended to the local case, as well. For briefness, we will not enter into such details.

The following statement, quoting [6] (see also [3,5,11]), gives a geometrical characterization of the graph of an isochronous potential: