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Lagrange multipliers in elastic–plastic torsion problem for nonlinear monotone operators

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Abstract

The existence of Lagrange multipliers as a Radon measure is ensured for an elastic–plastic torsion problem associated to a nonlinear strictly monotone operator. A regularization of this result, namely the existence of L^p Lagrange multipliers, is obtained under strong monotonicity assumption on the operator. Moreover, the relationships between elastic–plastic torsion problem and the obstacle problem are investigated. Finally, an example of the so-called "Von Mises functions" is provided, namely of solutions of the elastic–plastic torsion problem, associated to nonlinear monotone operators, which are not obtained by means of the obstacle problem in the case $f =$ constant.

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1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open bounded convex set with Lipschitz boundary $\partial\Omega$ and let $a(p) : \mathbb{R}^n \to \mathbb{R}^n$ be an operator of class C^2, strictly monotone, namely

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$$(a(P) - a(Q), P - Q) > 0 \quad \forall P, Q \in \mathbb{R}^n, P \neq Q. \tag{1}$$

We are aimed at the investigation of the existence of Lagrange multipliers associated to the following nonlinear problem

$$\text{Find } u \in K : \int_\Omega \sum_{i=1}^n a_i(Du) \left(\frac{\partial v}{\partial x_i} - \frac{\partial u}{\partial x_i} \right) dx \geq \int_\Omega f(v - u) dx, \quad \forall v \in K, \tag{2}$$

where

$$K = \left\{ v \in W_0^{1,\infty}(\Omega) : \sum_{i=1}^n \left(\frac{\partial v}{\partial x_i} \right)^2 \leq 1 \text{ a.e. on } \Omega \right\}.$$

In particular, we are able to prove that the Lagrange multiplier is always a Radon measure when the operator is strictly monotone, whereas the Lagrange multiplier is a L^p function when the operator is strongly monotone, namely there exists $\nu > 0$, such that

$$(a(P) - a(Q), P - Q) > \nu \|P - Q\|^2 \quad \forall P, Q \in \mathbb{R}^n, P \neq Q. \tag{3}$$

Our motivation for studying this type of problems is that these settings appear as the most natural and ultimate ones (existence and regularity of solutions are ensured in [2]).

The results of the paper have been achieved, using delicate tools of nonlinear partial differential equations and a new theory of infinite dimensional duality developed by the authors in [5–8,18–20], which has revealed itself effective also in nonlinear case. The classical theory of duality does not work in an infinite dimensional setting, when the interior of the ordering cone of the sign constraints is empty and this new theory overcomes this difficulty.

As in the linear case, also in the nonlinear case we are able to find the Von Mises functions, namely functions which clarify how the elastic–plastic solutions work.

Problem (2) is strictly related to the elastic–plastic torsion problem. According to Von Mises [28] (see also [21,25]), the elastic–plastic torsion problem of a cylindrical bar with cross section Ω is to find a function $\psi(x)$ which vanishes on the boundary $\partial\Omega$ and, together with its first derivatives, is continuous on Ω; nowhere on Ω the gradient of ψ must have an absolute value (modulus) less than or equal to a given positive constant τ; whenever in Ω the strict inequality holds, the function ψ must satisfy the differential equation $\Delta\psi = -2\mu\theta$, where the positive constants μ and θ denote the shearing modulus and the angle of twist per unit length respectively.

The elastic–plastic torsion problem and its relationships with obstacle problem have been deeply investigated in years 1965–1980 only for the Laplacian (see W. Ting [25–27] for $n = 2$ and H. Brezis [3] for the multidimensional case). Later on these studies have been resumed, with particular regards to existence and properties of Lagrange multipliers (see for linear operators [8,13,15,16,18,19], for generalized Lagrange multipliers [4]). Instead the results contained in the paper are concerned with the elastic–plastic torsion problem for nonlinear operators. In detail, when the operator is strictly monotone, we show that there exists $\bar{\mu} \in (L^\infty(\Omega))^*$ such that