Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 259 (2015) 989–1013

Journal of Differential Equations

www.elsevier.com/locate/jde

Bifurcation of the separatrix skeleton in some 1-parameter families of planar vector fields

Magdalena Caubergh

Departament de Matemàtiques, Edifici C, 08193 Bellaterra (Barcelona), Spain

Received 28 October 2014; revised 13 February 2015

Available online 5 March 2015

Abstract

This article deals with the bifurcation of polycycles and limit cycles within the 1-parameter families of planar vector fields X_m^k, defined by $\dot{x} = y^3 - x^{2k+1}$, $\dot{y} = -x + my^{4k+1}$, where m is a real parameter and $k \geq 1$ is an integer. The bifurcation diagram for the separatrix skeleton of X_m^k in function of m is determined and the one for the global phase portraits of $(X_m^1)_{m\in\mathbb{R}}$ is completed. Furthermore for arbitrary $k \geq 1$ some bifurcation and finiteness problems of periodic orbits are solved. Among others, the number of periodic orbits of X_m^k is found to be uniformly bounded independently of $m \in \mathbb{R}$ and the Hilbert number for $(X_m^k)_{m\in\mathbb{R}}$, that thus is finite, is found to be at least one.

MSC: 34C23; 34C07; 37G10; 37G15

Keywords: Rotated vector field; Global phase portrait; Separatrix skeleton; Limit cycle; Hilbert's 16th problem; Nilpotent center problem

1. Introduction

This article concerns periodic orbits and separatrix cycles for the 1-parameter families $(X_m^k)_{m\in\mathbb{R}}$, where X_m^k are planar polynomial vector fields of degree $4k+1$, given by

$$\dot{x} = y^3 - x^{2k+1}, \ \dot{y} = -x + my^{4k+1} \tag{1}$$

E-mail address: leen@mat.uab.cat.

http://dx.doi.org/10.1016/j.jde.2015.02.036
0022-0396/

depending on the parameter $m \in \mathbb{R}$, for arbitrary but fixed $k \geq 1$. Here both the nilpotent center-focus problem as well as the existential part of Hilbert's sixteenth problem for $(X_m^k)_{m\in\mathbb{R}}$ are approached.

The study of the particular family (1) is motivated by the questions raised in [11–13]. The authors in these papers presumed that the change of stability of the focus of (1) announces the birth of a connection between the two saddles. In this paper this presumption is confirmed qualitatively. Besides system (1) is a simple mathematical model whose study is not trivial and it gives the opportunity to illustrate a whole arsenal of methods classically used in the field. The next theorem summarizes the results from [11–13].

Theorem 1. *(See [13].) Let X_m^1 be defined by (1). For $m \leq 0$ the origin is a global attractor for X_m^1. For $m > 0$ the global phase portrait of X_m^1 is topologically equivalent to one of the four drawn in Fig. 3; in particular,*

1. *There are three singularities: a nilpotent focus at $(0, 0)$, which is stable for $0 < m < 3/5$ and unstable for $m \geq 3/5$, and two hyperbolic saddle points at $\mathbf{p}_\pm \equiv \mathbf{p}_\pm(m) = (\pm m^{-1/4}, \pm m^{-1/4})$.*
2. *For $m < 547/1000$ or $m \geq 3/5$ neither limit cycles nor polycycles do exist.*
3. *For $547/1000 \leq m < 3/5$ at most one limit cycle and polycycle exist and both cannot coexist. The limit cycle, if it exists, is hyperbolic and unstable. There exist $n \in \mathbb{N}$, $547/1000 < m_C^1 < \ldots < m_C^n < 3/5$ such that for $m = m_C^j$, $1 \leq j \leq n$, a heteroclinic 2-saddle cycle is formed.*

From numerical simulations the authors of [13] presumed that there is exactly one parameter value m_C for which X_m^1 presents a 2-saddle cycle. However the authors emphasize that a rigorous proof for its unicity is missing.

This article provides with an analytic confirmation of the unicity (see Theorem 5) and the bifurcation diagram of global phase portraits of X_m^1, $m > 0$ can thus be completed. Furthermore, here the case $k \geq 2$ is considered.

For $k \geq 2$ the bifurcation diagram of global phase portraits for $(X_m^k)_{m\in\mathbb{R}}$ is completed up to configurations of limit cycles of X_m^k. The analyses involves the control of separatrix and limit cycles, which are of global nature and therefore difficult to trace.

Recently, in [14], a technique is developed to localize separatrix bifurcations, which is applied in [15] to give fine estimates for the Bogdanov–Takens separatrix cycle. This technique does not apply for the family $(X_m^k)_m$. However the family transforms into a semi-complete family of indefinitely rotated vector fields $X_m^{k,R}$. Then the existence of the 2-saddle cycle is obtained from the behavior of the limit vector fields, both being strip flows with an algebraic curve of singularities. This argument differs from the one applied in [13] for the case $k = 1$, where one relies on Poincaré–Bendixson Theorem and limit cycle results. Next the uniqueness is proven exploiting the principles of the rotated property owned by $X_m^{k,R}$. Of course the monotonic movement is not necessary conserved by the separatrices of X_m^k. Nevertheless this has no influence on the bifurcation of the separatrix skeleton of X_m^k, $m > 0$.

In this article, for all $k \geq 1$, the relative movement of the separatrices at the hyperbolic saddles of X_m^k is controlled with increasing $m > 0$ and the bifurcation diagram for the separatrix skeleton of X_m^k with varying m thus is obtained (see Theorem 3). Furthermore, the absence of limit cycles is proven for m sufficiently small and m sufficiently large, that permits to apply the Roussarie