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The interior gradient estimate of Hessian quotient equations

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Abstract

In this paper, we establish the interior gradient estimate of k-admissible solutions of Hessian quotient equations $\frac{\sigma_k(D^2u)}{\sigma_l(D^2u)} = f(x, u, Du)$ with $0 \le l < k \le n$, which generalizes the result of [1,7].

MSC: 35J60; 35B45

Keywords: Hessian quotient equation; The interior gradient estimate; k-admissible solution

1. Introduction

In this paper, we consider the k-admissible solution of the Hessian quotient equation

$$\frac{\sigma_k(D^2u)}{\sigma_l(D^2u)} = f(x, u, Du), \quad x \in B_r(0) \subset \mathbb{R}^n, \tag{1.1}$$

with $0 \le l < k \le n$. For any $k = 1, \cdots, n$,

$$\sigma_k(D^2u) = \sigma_k(\lambda(D^2u)) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k},$$

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with $\lambda(D^2u) = (\lambda_1, \cdots, \lambda_n)$ being the eigenvalues of D^2u. We also set $\sigma_0 = 1$. And we recall that the Garding cone is defined as

$$\Gamma_k = \{\lambda \in \mathbb{R}^n : \sigma_i(\lambda) > 0, \forall 1 \le i \le k\}.$$

If $\lambda(D^2u) \in \Gamma_k$ for any $x \in B_r(0)$, then Eq. (1.1) is elliptic (see [5]), and we say u is a k-admissible solution of (1.1). In particular, (1.1) is the Laplacian equation if $k = 1, l = 0$, and the Monge–Ampere equation if $k = n, l = 0$.

To my best knowledge, there is no a priori estimate for the Hessian quotient equation (1.1) in Euclidean space, but there is a result in conformal geometry [2]. In this paper, we establish the interior gradient estimate as follows,

Theorem 1.1. *Suppose $u \in C^3(B_r(0))$ is a k-admissible solution to the Hessian quotient equation (1.1), with $f(x, u, Du) > 0$ in $B_r(0)$ and $f(x, u, p) \in C^1(B_r(0) \times \mathbb{R} \times \mathbb{R}^n)$. Then*

$$|Du(0)| \le C\Big(\frac{\operatorname*{osc}_{B_r(0)} u}{r} + [\operatorname*{osc}_{B_r(0)} u]^{\frac{k-l+1}{2(k-l)}} + [\operatorname*{osc}_{B_r(0)} u]^{\frac{k-l}{2(k-l)+1}}\Big), \tag{1.2}$$

where C is a positive constant depending only on n, k, l and $|D_x f|_{C^0}$, $|D_u f|_{C^0}$, $|D_p f|_{C^0}$. In particular, if $f =$ constant > 0, we have

$$|Du(0)| \le C\frac{\operatorname*{osc}_{B_r(0)} u}{r}, \tag{1.3}$$

where C is a positive constant depending only on n, k, l.

Remark 1.2. The estimate does not depend on $|f|_{C^0}$ and $\inf f$. Moreover, if $\operatorname*{osc}_{B_r(0)} u$ is very small, the gradient $Du(0)$ is also very small.

Remark 1.3. When $l = 0$, the corresponding results are due to [1,7].

Remark 1.4. Similarly to [1,8], we can get the corresponding gradient estimates for the parabolic Hessian quotient equation.

The rest of the paper is organized as follows. In Section 2, we collect some properties of σ_k, which will be used in the proof of Theorem 1.1. In Section 3, we prove Theorem 1.1, following the idea of [1].

2. Preliminary

In this section, we recall the definition and some basic properties of elementary symmetric functions, which could be found in [5].

First, we denote by $\sigma_k(\lambda\,|\,i)$ the symmetric function with $\lambda_i = 0$ and $\sigma_k(\lambda\,|\,ij)$ the symmetric function with $\lambda_i = \lambda_j = 0$.

Proposition 2.5. *Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ and $k = 0, 1, \cdots, n$, then*