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Dirichlet problems with singular convection terms and applications

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A Guido ("'l mio autore"[1]), a Thierry ("'l mio co-autore"[2])

Abstract

In this paper, we study Dirichlet problems with singular convection terms. Then applications to the study of some elliptic systems of two equations are given.

1. Introduction

Several years ago, Thierry Gallouët sent me his paper [17], devoted to the mathematical analysis of models of flows in porous media. The reading drew my attention to the mathematical framework which is the subject of the present paper. The key point is the presence of a term of the type $\mathrm{div}(uE)$, where u is the solution and E is a vector field with L^2 summability.

My first approach to this type of problems was at the University of Roma, during the academic year 1969–1970, when Guido Stampacchia was my teacher for the "Analisi Superiore" course. In a recent paper [4], dedicated to the memory of him on the thirtieth anniversary of his death, I improved some of his results (see [20]) concerning the Dirichlet problem

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[1] (Dante: Inferno I.)

[2] For his birthday.

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$$\begin{cases} -\operatorname{div}(M(x)\nabla u) = -\operatorname{div}(uE(x)) + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Here Ω is a bounded, open subset of $\mathbb{R}^N$, $N > 2$,

$$E \in (L^N(\Omega))^N, \tag{2}$$

$$f \in L^m(\Omega), \quad 1 \le m < \frac{N}{2}, \tag{3}$$

and $M(x)$ is a bounded and measurable matrix such that

$$\alpha|\xi|^2 \le M(x)\xi\xi, \qquad |M(x)| \le \beta, \quad \text{a.e. } x \in \Omega, \ \forall \xi \in \mathbb{R}^N. \tag{4}$$

To be more precise, in [4] the existence of u is proved

$$\begin{cases} \text{weak solution belonging to } W_0^{1,2}(\Omega) \cap L^{m^{**}}(\Omega), & \text{if } m \ge \frac{2N}{N+2}; \\ \text{distributional solution belonging to } W_0^{1,m^*}(\Omega), & \text{if } 1 < m < \frac{2N}{N+2}; \end{cases} \tag{5}$$

where $m^* = \frac{mN}{N-m}$ $(1 \le m < N)$ and $m^{**} = \frac{mN}{N-2m}$ $(1 \le m < \frac{N}{2})$. Note that the above existence results are exactly the results proved with $E = 0$ in [20] and [8].

Thus the roots of this paper are Guido Stampacchia and Thierry Gallouët (see also the first page of [3]), as well as some techniques coming from papers by the first one, and from papers in collaboration with the second one.

1.1. Setting

In the present paper, differential problems with vector fields E which do not belong to $(L^N(\Omega))^N$ are considered. An important feature on E is that we do not make any assumption on the divergence of the field E (see Remark 3.4). The most important aim (both for the importance and the difficulty) is the study of the case $E \in (L^2(\Omega))^N$, where a key point is the definition of solution. It is possible to give a meaning to solution for problem (1), using the concept of *entropy solutions* introduced in [1]. In order to use the functional framework of [1] an important point is the observation that even u does not belong to $W_0^{1,2}(\Omega)$, where u is a solution, nevertheless $\|T_k(u)\|_{W_0^{1,2}(\Omega)} \le C(k)$, where T_k is the truncature at levels $\pm k$ and $C(k)$ is an unbounded function of k. A starting point for the role of the truncature, in problems with infinite energy solutions, is the paper [9] (see also [2,5]). Other works related with this paper are [13,10,22–24].

Even if $E \in (L^N(\Omega))^N$, a difficulty is due to the noncoercivity in $W_0^{1,2}(\Omega)$ of the differential operator $-\operatorname{div}(M(x)\nabla v) + \operatorname{div}(vE(x))$.

As in [4], we use a nonlinear approach to the linear noncoercive boundary value problem (1): we consider the following approximate Dirichlet problems

$$u_n \in W_0^{1,2}(\Omega) : -\operatorname{div}(M(x)\nabla u_n) = -\operatorname{div}\left(\frac{u_n}{1 + \frac{1}{n}|u_n|} \frac{E(x)}{1 + \frac{1}{n}|E(x)|}\right) + f_n(x), \tag{6}$$

where