Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 258 (2015) 2364–2374
Journal of Differential Equations
www.elsevier.com/locate/jde

Convexity of the solutions to the constant mean curvature spacelike surface equation in the Lorentz–Minkowski space

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Received 21 July 2014
Available online 29 December 2014

Abstract

We prove that a spacelike graph of constant mean curvature $H \neq 0$ in the 3-dimensional Lorentz–Minkowski space over a bounded domain with pseudo-elliptic boundary is strictly convex. By a pseudo-elliptic curve we mean a closed and planar curve which intersects any branch of any hyperbola at most at five points. We also provide an example that shows that we cannot remove the assumption on the boundary being a pseudo-elliptic curve.

MSC: 35J93; 53C42; 53C50

Keywords: Spacelike hypersurface; Constant mean curvature; Dirichlet problem; Convex curve; Strictly convex surface

1. Introduction

A hypersurface in the Lorentz–Minkowski space $\mathbb{L}^n$ is called spacelike if its induced metric from $\mathbb{L}^n$ is Riemannian. Spacelike hypersurfaces of constant mean curvature (CMC) are critical points of the area functional under a suitable volume constraint [1,2]. Such hypersurfaces play

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http://dx.doi.org/10.1016/j.jde.2014.12.011
0022-0396/

an important role in general relativity, since they can be used as initial data where the constraint equations can be split into a linear system and a nonlinear elliptic equation (see [3] and references therein). A summary of other reasons justifying the study of CMC spacelike hypersurfaces can be found in [4].

Along this paper we will work with compact CMC spacelike surfaces immersed in $\mathbb{L}^3$ with (necessarily) non-empty smooth boundary, which will be supposed to be planar and simple. We will study the influence of the geometry of the boundary on the shape of the surface. All those surfaces are known to be graphs over a spacelike plane, see [5, Corollary 12.1.8], in contrast with what happens in the Euclidean space $\mathbb{R}^3$. Up to an isometry, one can assume the plane to be the $z = 0$ plane. Specifically, if Ω is a domain of $\mathbb{R}^2$, then every smooth function $u \in \mathcal{C}^\infty(\Omega)$ determines a graph over Ω given by

$$\Sigma_u = \big\{\big(x, y, u(x, y)\big) : (x, y) \in \Omega\big\} \subset \mathbb{L}^3.$$

The graph Σ_u is a CMC spacelike surface with boundary $\partial\Omega$ if and only if the function u is the solution of the Dirichlet problem

$$\begin{aligned} \operatorname{div}\left(\frac{Du}{\sqrt{1-|Du|^2}}\right) &= 2H, \quad |Du| < 1 && \text{in } \Omega, \\ u &= 0 && \text{along } \partial\Omega, \end{aligned}$$

where H is a constant and D, div and $|\cdot|$ stand for the gradient and divergence operators, and the norm in the Euclidean plane $\mathbb{R}^2$, respectively (see Section 2 for the details).

The existence and uniqueness of a function $u \in \mathcal{C}^2(\Omega) \cup \mathcal{C}^0(\bar{\Omega})$ being the solution of the previous problem are always assured for any bounded domain Ω, see [6] and [5, Theorem 12.2.2]. There are two particular cases in which the solution is known. When $H = 0$, u must be zero too. When $H \neq 0$ and Ω is a round disc of radius R (which is supposed to be centered at the origin), then

$$u(x, y) = \sqrt{x^2 + y^2 + \frac{1}{H^2}} - \sqrt{R^2 + \frac{1}{H^2}},$$

that is, Σ_u is a hyperbolic cap [7].

In contrast with the Lorentzian case, the existence of the Dirichlet problem for the CMC equation in the Euclidean space is not assured and it depends on the domain Ω. In this sense, a classical result due to Serrin [8] establishes the existence and uniqueness of a solution to the Dirichlet problem for any given boundary condition if and only if the curvature κ of $\partial\Omega$ satisfies $\kappa \geq 2H \geq 0$.

Another significant difference appears when estimating the height of the solution. While $1/|H|$ is a bound in the Euclidean case with zero boundary condition, see [9], the only possible bounds in the Lorentzian case involve either the diameter of Ω [5, Corollary 12.4.6], or the area of the graph [10].

As we have mentioned, we are interested in studying the influence of the geometry of the boundary on Σ_u. In particular, we ask if the convexity of $\partial\Omega$ is inherited by Σ_u. It is a classical problem to study if the convexity of the domain of a boundary value problem associated to an elliptic partial differential equation implies convexity of the solution [11–16]. In this paper we consider bounded domains whose boundary curve is planar, with the added property that