Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 258 (2015) 2618–2632

Journal of Differential Equations

www.elsevier.com/locate/jde

On the regularity of the free boundary in the optimal partial transport problem for general cost functions

S. Chen [a], E. Indrei [b,*]

[a] *Department of Mathematics, Zhejiang University of Technology, Hangzhou 310023, China*
[b] *Center for Nonlinear Analysis, Carnegie Mellon University, Pittsburgh, PA 15213, USA*

Received 9 December 2013; revised 4 December 2014

Available online 31 December 2014

Abstract

This paper concerns the regularity and geometry of the free boundary in the optimal partial transport problem for general cost functions. More specifically, we prove that a C^1 cost implies a locally Lipschitz free boundary. As an application, we address a problem discussed by Caffarelli and McCann [1] regarding cost functions satisfying the Ma–Trudinger–Wang condition (A3): if the non-negative source density is in some $L^p(\mathbb{R}^n)$ space for $p \in (\frac{n+1}{2}, \infty]$ and the positive target density is bounded away from zero, then the free boundary is a semiconvex $C^{1,\alpha}_{loc}$ hypersurface. Furthermore, we show that a locally Lipschitz cost implies a rectifiable free boundary and initiate a corresponding regularity theory in the Riemannian setting.
© 2014 Elsevier Inc. All rights reserved.

1. Introduction

In the optimal partial transport problem, one is given two non-negative functions $f = f\chi_\Omega$, $g = g\chi_\Lambda \in L^1(\mathbb{R}^n)$ and a number $0 < m \le \min\{\|f\|_{L^1}, \|g\|_{L^1}\}$. The objective is to find an optimal transference plan between f and g with mass m. A transference plan refers to a non-negative, finite Borel measure γ on $\mathbb{R}^n \times \mathbb{R}^n$, whose first and second marginals are controlled by f and g respectively: for any Borel set $A \subset \mathbb{R}^n$,

* Corresponding author.
E-mail addresses: schen@zjtu.edu.cn (S. Chen), egi@cmu.edu (E. Indrei).

http://dx.doi.org/10.1016/j.jde.2014.12.016
0022-0396/© 2014 Elsevier Inc. All rights reserved.

$$\gamma\big(A\times\mathbb{R}^n\big)\le\int\limits_A f(x)dx,\qquad \gamma\big(\mathbb{R}^n\times A\big)\le\int\limits_A g(x)dx.$$

An optimal transference plan is a minimizer of the functional

$$\gamma\mapsto\int\limits_{\mathbb{R}^n\times\mathbb{R}^n} c(x,y)d\gamma(x,y), \tag{1.1}$$

where c is a non-negative cost function.

Issues of existence, uniqueness, and regularity of optimal transference plans have recently been addressed by Caffarelli and McCann [1], Figalli [2,3], and Indrei [4]. Indeed, existence follows readily by standard methods in the calculus of variations. However, in general, minimizers fail to be unique and it is not difficult to construct examples when $|\mathrm{spt}(f\wedge g)|>0$ (with $|\cdot|$ being the Lebesgue measure and $\mathrm{spt}(f\wedge g)$ the support of $f\wedge g:=\min\{f,g\}$). Nevertheless, Figalli proved that under suitable assumptions on the cost function, minimizers are unique for

$$\|f\wedge g\|_{L^1(\mathbb{R}^n)}\le m\le\min\big\{\|f\|_{L^1(\mathbb{R}^n)},\|g\|_{L^1(\mathbb{R}^n)}\big\},$$

[2, Proposition 2.2 and Theorem 2.10]. Up to now, the regularity theory has only been developed for the quadratic cost. In this case, if the domains Ω and Λ are bounded, strictly convex, and separated by a hyperplane, Caffarelli and McCann proved (under suitable conditions on the initial data) that the free boundaries $\overline{\partial U_m\cap\Omega}$ and $\overline{\partial V_m\cap\Lambda}$ are locally $C^{1,\alpha}$ hypersurfaces up to a closed singular set $\tilde{S}$ contained at the intersection of free with fixed boundary [1, Corollary 7.15]; here, the free boundaries are generated by the sets U_m and V_m which are referred to as the "active regions." U_m is defined as the interior of the support of the left marginal of the optimal transference plan, and V_m is similarly defined in terms of the right marginal (a characterization of these regions in terms of the cost function is given by [1, Corollary 2.4]).

In the case when there is overlap, Figalli proved that away from the common region $\Omega\cap\Lambda$, the free boundaries are locally C^1 [2, Theorem 4.11]. Indrei improved this result by obtaining local $C^{1,\alpha}$ regularity away from the common region and up to a relatively closed singular set S, necessarily contained at the intersection of fixed with free boundary, see [4, Corollary 3.13] for a precise statement. Moreover, under an additional $C^{1,1}$ regularity assumption on Ω and Λ, he proved that S is $\mathcal{H}^{n-2}$ σ-finite and in the disjoint case $S\subset\tilde{S}$ with $\mathcal{H}^{n-2}(S)<\infty$ [4, Theorem 4.9].

All of the aforementioned regularity results were developed for the quadratic cost. Our main aim in this paper is to obtain free boundary regularity for a general class of cost functions $\mathcal{F}_0$ satisfying the Ma–Trudinger–Wang (A3) condition introduced in [8] and used in the development of a general regularity theory for the potential arising in the optimal transportation problem (see Definition 2.4). With this in mind, we establish the following theorem which readily implies $C^{1,\alpha}_{loc}$ regularity of the free boundary for the family $\mathcal{F}_0$ and thereby solves a problem discussed by Caffarelli and McCann [1, p. 676].

Theorem 1.1 *(Lipschitz regularity). Let $f=f\chi_\Omega$, $g=g\chi_\Lambda$ be non-negative integrable functions and $m\in(0,\min\{\|f\|_{L^1},\|g\|_{L^1}\}]$. Assume that Λ is bounded and c-convex with respect to Ω, where $c\in C^1(\mathbb{R}^n\times\mathbb{R}^n)$ and satisfies (2.1) and (2.2). Then the free boundaries arising in the optimal partial transport problem are locally Lipschitz graphs inside Ω.*

Download English Version:

https://daneshyari.com/en/article/4610073

Download Persian Version:

https://daneshyari.com/article/4610073

Daneshyari.com