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Existence and uniqueness of solutions for a boundary value problem arising from granular matter theory

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Abstract

We consider a system of PDEs of Monge–Kantorovich type that, in the isotropic case, describes the stationary configurations of two-layers models in granular matter theory with a general source and a general boundary data. We propose a weak formulation which is consistent with the physical model and permits us to prove existence and uniqueness results.

MSC: primary 35A02; secondary 35J25

Keywords: Boundary value problems; Mass transfer theory

1. Introduction

The model system usually considered for the description of the stationary configurations of sandpiles on a container is the Monge–Kantorovich type system of PDEs

$$\begin{cases} -\operatorname{div}(v\,Du) = f & \text{in } \Omega, \\ |Du| \le 1, \ \ v \ge 0 & \text{in } \Omega, \\ (1-|Du|)v = 0 & \text{in } \Omega, \\ u \le \phi & \text{on } \partial\Omega, \\ u = \phi & \text{on } \Gamma_f \end{cases} \tag{1}$$

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(see, e.g., [2,4,6,15]). The data of the problem are the flat surface of the container $\Omega \subseteq \mathbb{R}^2$, the profile of the rim ϕ, and the density of the source $f \geq 0$, whereas the set Γ_f is a subset of $\partial\Omega$, defined in terms of the other data, that will be specified below.

The dynamical behavior of the granular matter is pictured by the pair (u, v), where u is the profile of the standing layer, whose slope has not to exceed a critical value ($|Du| \leq 1$) in order to prevent avalanches, while $v \geq 0$ is the thickness of the rolling layer. The condition $(1 - |Du|)v = 0$ corresponds to require that the matter runs down only in the region where the slope of the heaps is maximal.

The set Γ_f (which depends on the source f, the geometry of Ω, and on the boundary datum ϕ) is the part of the border where every admissible profile u touches the rim, in such a way the exceeding sand can fall down (see Definitions in Section 3). We underline that the set Γ_f is not an additional datum of the problem, but it is constructed in terms of the other data (see (9) for its precise definition).

The main contribution of our results to the theory concerns the uniqueness of the v-component for general boundary value problems, based on a new weak formulation of the continuity equation $-\operatorname{div}(v\, Du) = f$ in Ω.

The case of the open table problem, corresponding to $u = \phi = 0$ on $\partial\Omega$, is already completely understood (see e.g. [4–6,17] and the references therein). Namely, if d_Ω denotes the distance function from the boundary of Ω, it is possible to construct a function $v_f \geq 0$, $v_f \in L^1(\Omega)$ such that the pair (d_Ω, v_f) is a solution to (1) (we underline that the continuity equation is understood in the sense of distributions). Moreover it turns out that v_f is the unique admissible v-component, and every profile u must coincides with d_Ω where the transport is active.

These results validate the model for the open table problem, since they depicted the sole physically acceptable situation: the mass transport density v has to be uniquely determined by the data of the problem, while the profile u could be different from the maximal one only where the mass transportation does not act.

Moreover the profile is unique (and maximal) if and only if the source f pours sand along the ridge of the maximal profile (i.e. on the closure of the set where d_Ω is not differentiable).

As far as we know, only the following two particular cases of non-homogeneous boundary conditions were considered in literature.

In [13], mostly devoted to a numerical point of view, the problem of the open table with walls (corresponding to $u = \phi = 0$ on a regular portion Γ of $\partial\Omega$, and $\phi = +\infty$ in $\partial\Omega \setminus \Gamma$) is considered. In order to take into account the fact that the sand can flow out from the table only through Γ, the weak formulation of the continuity equation proposed in [13] is the following:

$$\int_\Omega v\langle Du,\, D\psi\rangle\, dx = \int_\Omega f\psi\, dx, \qquad \forall \psi \in C_c^\infty(\mathbb{R}^2 \setminus \bar{\Gamma}).$$

Under suitable regularity assumptions on the geometry of the sandpile, it is proved that there exists a function $v_f \geq 0$, $v_f \in L^1(\Omega)$ such that the pair (d_Γ, v_f) is a solution to (1), where d_Γ is the distance function from Γ.

A different approach to non-homogeneous boundary conditions was recently proposed in [12]. In that paper we considered only admissible boundary data, that is continuous functions ϕ on $\partial\Omega$ that coincide on the boundary with the related Lax–Hopf function u_ϕ. In the model, this corresponds to treating the so called tray table problem, where the boundary datum ϕ gives the height of the rim. The requirements are that the border of the rim is always reached ($u = \phi$