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Bifurcation of critical periods of polynomial systems

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Abstract

We describe a general approach to studying bifurcations of critical periods based on a complexification of the system and algorithms of computational algebra. Using this approach we obtain upper bounds on the number of critical periods of several families of cubic systems. In some cases we overcome the problem of nonradicality of a relevant ideal by moving it to a subalgebra generated by invariants of a group of linear transformations.

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1. Introduction

Consider a system of ordinary differential equations on $\mathbb{R}^2$ of the form

$$\dot{u} = -v + P(u, v), \ \ \dot{v} = u + Q(u, v), \tag{1}$$

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where u and v are real variables and P and Q are polynomials without constant or linear terms. The singularity of system (1) at the origin is either a center or a focus. When it is a center we can consider the so-called period function $T(r)$, which gives the least period of the periodic solution passing through the point with coordinates $(u, v) = (r, 0)$ inside the period annulus. A center is *isochronous* if all nonconstant solutions in a neighborhood have the same period. For a center that is not isochronous any value $r > 0$ for which $T'(r) = 0$ is called a *critical period*.

Properties of period functions have been extensively studied. For example monotonicity properties were investigated in [3,5,8–10,16,27,35] and finitude of critical periods in [6,26]. An extensive literature is devoted to isochronicity of centers in systems (1), see for example [3,19,21] and the references they contain.

The problem of interest for us, the so-called problem of critical period bifurcations, was considered for the first time by Chicone and Jacobs [7]. The problem is to estimate the number of critical periods that can arise near the center under small perturbation of system (1) within some family that contains it. In [7] this problem was studied for quadratic systems and some Hamiltonian systems. Bifurcations of critical periods for a linear center perturbed by homogeneous cubic polynomials were investigated in [19,31]. The problem has also been studied for reversible cubic systems [38], the reduced Kukles system [32], Liénard systems [40], generalized Lotka–Volterra systems [36], third-order planar Hamiltonian systems [37], generalized Loud systems [34], and reversible rigidly isochronous centers [4,23,39].

In all previous work devoted to critical period bifurcations the authors parametrized each component of the center variety (the subset of parameter space corresponding to centers) and derived, often laboriously, an upper bound on the number of critical periods bifurcating from each component. In this paper we describe an approach that allows one to obtain an upper bound on the number of critical periods for all components of the center variety simultaneously. Our approach is based on complexifying the real family and applying methods of computational commutative algebra and is algorithmic, provided the center and linearizability problems have already been solved. It can be applied to any family of polynomial systems for which the center and linearizability problems have been solved and for which the computations are feasible on available computational facilities.

After two sections of preliminary and background material we present the method in Section 4, formulated as Theorem 4.1, and illustrate it by obtaining in Theorem 4.4 an upper bound of two for the number of critical periods that can bifurcate from a center at the origin of the real family whose expression in complex form (see (7)) is

$$\dot{x} = i(x + ax^2 + bx^3 + cx^2\bar{x} + dx\bar{x}^2), \tag{2}$$

selected because the center and linearizability varieties are known for this family and because a certain ideal in the polynomial ring over $\mathbb{C}$ in the parameters of the family is radical. The method we describe works well when this ideal is radical. When it is not radical it is possible to obtain a bound for a restricted set of centers by working separately with prime and non-prime components in a primary decomposition of the ideal of interest. This extension of the method is described in Section 5, where it is formulated as Theorem 5.2. It is illustrated in Theorem 5.6, which states that for the full family in (1) with only homogeneous cubic nonlinearities, whose expression in complex form is

$$\dot{x} = i(x + ax^3 + bx^2\bar{x} + cx\bar{x}^2 + d\bar{x}^3), \tag{3}$$