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On the uniqueness of solutions to continuity equations

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Abstract

We obtain sufficient conditions for the uniqueness of solutions to the Cauchy problem for the continuity equation in classes of measures that need not be absolutely continuous.

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0. Introduction

In this paper we study the uniqueness problem for the continuity equation

$$\partial_t \mu_t + \operatorname{div}(b\mu_t) = 0$$

with respect to measures on $\mathbb{R}^d$. We consider solutions given by families of locally bounded Borel measures $(\mu_t)_{t\in[0,T]}$. A precise definition is given below.

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There is a vast literature devoted to uniqueness and existence problems for the Cauchy problem for such equations. An important problem is to specify a class of measures μ_t in which, under reasonable assumptions about coefficients and initial data, there is a unique solution to the Cauchy problem. Certainly, if the coefficients are sufficiently regular, say, Lipschitzian or satisfy the Osgood type condition, then we can take the whole class of bounded measures μ_t (see, e.g., [4]). According to a well-known result of Ambrosio [3] (see also [30] for equations with a potential term) on representations of nonnegative bounded solutions by means of averaging with respect to measures concentrated on solutions to the corresponding ordinary equation $\dot{x} = b(x, t)$, any uniqueness condition for the ordinary equation guarantees uniqueness in the class of nonnegative bounded measures. However, in the class of signed measures there is no such representation.

In the case of non-smooth coefficients a class convenient in many respects is the class of measures absolutely continuous with respect to Lebesgue measure, which is quite natural, in particular, taking into account existence results. A study of this class initiated by Cruzeiro [18–20] and DiPerna and Lions [25] was continued by many researchers. A large number of papers are devoted to the so-called Lagrangian flows and their generalizations (see [2] and [3]). However, this class of absolutely continuous measures is rather narrow, in particular, it does not enable one to deal with singular initial data (and is essentially oriented towards vector fields having at least some minimal regularity such as the existence of their divergence or being BV, see [32]). In addition, this class has no universal analogs in infinite dimensions. There are several papers (see [18,33,11,35,2,21,28]) concerned with the infinite-dimensional case and using as reference measures certain special measures (all reducing to the absolute continuity with respect to Lebesgue measure when the infinite-dimensional state space is replaced by $\mathbb{R}^d$) such as Gaussian measures, convex measures, and differentiable measures, which becomes rather restrictive in infinite dimensions in spite of importance of such classes of measures in applications (e.g., when they are Gibbs measures). The recent paper [8] develops continuity equations in metric measure spaces, but again only considering solutions absolutely continuous with respect to the underlying fixed measure.

Thus, it is natural to look for other classes of measures, apart from absolutely continuous measures, in which the existence and uniqueness of solutions hold in the case of non-smooth coefficients. In this paper we consider the finite-dimensional case; it turns out that even in the one-dimensional case in the present framework new results can be obtained.

The main result in this direction obtained in our paper can be briefly formulated as follows: uniqueness holds in a certain class of measures with respect to which the given vector field b can be suitably approximated by smooth vector fields (thus, the uniqueness class may depend on b). A precise formulation is given below (see Theorems 1.1 and 1.4), but we observe that this result is consistent with typical methods of constructing solutions when b is approximated by smooth fields b_k and the solution is obtained as a limit point of the sequence of solutions μ_t^k for b_k. As in many existing papers, our conditions admit discontinuous fields, but the hypotheses are mostly incomparable (with the already cited papers and, e.g., [1,5,6,13–15,17,22–24,26,27]).

As an application we obtain some new results for the continuity equation with a merely continuous coefficient b. In particular, we substantially improve the recent result from [16], where the uniqueness is proved in dimension one for absolutely continuous solutions under the assumptions that b is continuous and nonnegative, the trajectories of $\dot{x} = b(x)$ do not blow up in finite time and the set of zeros $Z = \{x : b(x) = 0\}$ consists of a finite union of points and closed intervals. For example, we prove the following assertion: if $b \in C(\mathbb{R})$ and $0 \le b(x) \le C + C|x|$ (which is a constructive condition to exclude a blowup), then the uniqueness holds in the class