Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 259 (2015) 3894–3928

Journal of Differential Equations

www.elsevier.com/locate/jde

Multiple normalized solutions for quasi-linear Schrödinger equations

Louis Jeanjean [a], Tingjian Luo [b,a,*], Zhi-Qiang Wang [c,d]

[a] *Laboratoire de Mathématiques (UMR 6623), Université de Franche-Comté, 16 Route de Gray, 25030 Besançon Cedex, France*

[b] *School of Mathematics and Information Science, Guangzhou University, Guangzhou 510006, PR China*

[c] *Center for Applied Mathematics, Tianjin University, Tianjin 300072, PR China*

[d] *Department of Mathematics and Statistic, Utah State University, Logan, UT84322, USA*

Received 30 November 2013; revised 23 April 2015

Available online 16 May 2015

Abstract

In this paper we prove the existence of two solutions having a prescribed L^2-norm for a quasi-linear Schrödinger equation. One of these solutions is a mountain pass solution relative to a constraint and the other one a minimum either local or global. To overcome the lack of differentiability of the associated functional, we rely on a perturbation method developed in [25].

MSC: 35J50; 35Q41; 35Q55; 37K45

Keywords: L^2-normalized solutions; Liouville type results; Quasi-linear Schrödinger equations; Perturbation method

1. Introduction

In this paper, we are concerned with quasi-linear Schrödinger equations of the form

$$\begin{cases} i\partial_t \varphi + \Delta\varphi + \varphi\Delta(|\varphi|^2) + |\varphi|^{p-1}\varphi = 0, & \text{in } \mathbb{R}^+ \times \mathbb{R}^N, \\ \varphi(0,x) = \varphi_0(x), & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

* Corresponding author.

E-mail addresses: louis.jeanjean@univ-fcomte.fr (L. Jeanjean), tingjianluo@gmail.com (T. Luo), zhi-qiang.wang@usu.edu (Z.-Q. Wang).

http://dx.doi.org/10.1016/j.jde.2015.05.008

where $p \in (1, \frac{3N+2}{N-2})$ if $N \geq 3$ and $p \in (1, \infty)$ if $N = 1, 2$, i denotes the imaginary unit and the unknown $\varphi : \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{C}$ is a complex valued function. Such types of equations appear in various physical fields, for instance in dissipative quantum mechanics, in plasma physics and in fluid mechanics. We refer the readers to [11,29] and their references for more information on the related physical backgrounds.

From the physical as well as the mathematical point of view, a central issue is the existence and dynamics of standing waves of (1.1). By standing waves, we mean solutions of the form $\varphi(t, x) = e^{-i\lambda t}u(x)$, where $\lambda \in \mathbb{R}$ is a parameter. Observe that $e^{-i\lambda t}u(x)$ solves (1.1) if and only if $u(x)$ satisfies the following stationary equation

$$-\Delta u - u\Delta(|u|^2) - \lambda u - |u|^{p-1}u = 0, \quad \text{in } \mathbb{R}^N. \tag{P_λ}$$

In (P_λ), when $\lambda \in \mathbb{R}$ appears as a fixed parameter, the existence and multiplicity of solutions of (P_λ) have been intensively studied during the last decade. See [4,8,10,11,13,20–26,29,30] and their references therein. We also refer to [3,4,15,31] for the uniqueness of ground states of (P_λ). By a ground state we mean a solution of (P_λ) which minimize among all nontrivial solutions the associated energy functional

$$I_\lambda(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{\lambda}{2}\int_{\mathbb{R}^N} |u|^2 dx + \int_{\mathbb{R}^N} |u|^2|\nabla u|^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1} dx,$$

defined on the natural space

$$\mathcal{X} := \left\{ u \in W^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2|\nabla u|^2 dx < \infty \right\}.$$

It is easy to check that u is a weak solution of (P_λ) if and only if

$$I'_\lambda(u)\phi := \lim_{t \to 0^+} \frac{I_\lambda(u + t\phi) - I_\lambda(u)}{t} = 0,$$

for every direction $\phi \in C_0^\infty(\mathbb{R}^N, \mathbb{R})$. We also recall, see [11, Remark 1.7] and [22] for example, that when $N \geq 3$ the value $\frac{3N+2}{N-2}$ corresponds to a critical exponent.

Compared to semi-linear equations where the term $u\Delta(|u|^2)$ is not present, the search of solutions of (P_λ) presents a major difficulty. The functional associated with the quasi-linear term

$$V(u) := \int_{\mathbb{R}^N} |u|^2|\nabla u|^2 dx,$$

is non differentiable in the space $\mathcal{X}$ when $N \geq 2$. To overcome this difficulty, various arguments have been developed. First in [20,29], solutions of (P_λ) are obtained by minimizing the functional I_λ on the set

$$\left\{ u \in \mathcal{X} : \int_{\mathbb{R}^N} |u|^{p+1} dx = 1 \right\}.$$