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On uniqueness of weak solutions for the thin-film equation

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Abstract

In any number of space variables, we study the Cauchy problem related to the fourth-order degenerate diffusion equation $\partial_s h + \nabla \cdot (h\nabla\Delta h) = 0$. This equation, derived from a lubrication approximation, also models the surface tension dominated flow of a thin viscous film in the Hele-Shaw cell. Our focus is on uniqueness of weak solutions in the "complete wetting" regime, when a zero contact angle between liquid and solid is imposed. In this case, we transform the problem by zooming into the free boundary and look at small Lipschitz perturbations of a quadratic stationary solution. In the perturbational setting, the main difficulty is to construct scale invariant function spaces in such a way that they are compatible with the structure of the nonlinear equation. Here we rely on the theory of singular integrals in spaces of homogeneous type to obtain linear estimates in these functions spaces which provide "optimal" conditions on the initial data under which a unique solution exists. In fact, this solution can be used to define a class of functions in which the original initial value problem has a unique (weak) solution. Moreover, we show that the interface between empty and wetted regions is an analytic hypersurface in time and space.

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1. Introduction

1.1. The model

In this paper, we will be discussing the Cauchy problem related to the fourth-order degenerate equation

$$\partial_s h + \nabla \cdot \left(h^m \nabla \Delta h\right) = 0. \tag{tfe}$$

Here $h : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ is a real-valued function of time and space, the gradient and the Laplacian are in the space variables only, and $m > 0$.

In a setting where the height of the liquid is small compared to its horizontal propagation, we can employ a lubrication approximation[1] of the governing Navier–Stokes equations, i.e. a limit for vanishing $\varepsilon =$ height/length, to get

$$\mu\, \partial_z^2 v = -\gamma\, \nabla \Delta h,$$

relating the fluid's velocity in the horizontal directions to its thickness and shape at the liquid–vapor interface. Here the constants μ and γ denote the viscosity and the surface tension, respectively. The variable $z \in (0, h)$ is the coordinate perpendicular to the surface. From the classical theory of fluids we learn that when a fluid is put onto a solid ground, the correct slippage model is a no-slip condition at the liquid–solid interface. This, however, would lead to a singularity in the dissipation for propagating contact lines, an observation that was first made by Huh and Scriven [17]. In order to remove this singularity, one usually fixes the flow condition on the liquid–solid interface by

$$v = b^{3-m} h^{m-2} \partial_z v \quad \text{for} \quad z = 0,$$

[1] Ref. [25] and the references therein provide a comprehensive discussion on this.