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BV continuous sweeping processes

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Abstract

We consider a large class of continuous sweeping processes and we prove that they are well posed with respect to the BV strict metric.

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1. Introduction

Let $\mathcal{X}$ be real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and let $\mathcal{C}(t) \subseteq \mathcal{X}$ be a family of nonempty closed convex sets parametrized by the time variable $t \in [0, T]$, where $T > 0$. A *sweeping process* is the following evolution differential inclusion in the unknown $\xi : [0, T] \longrightarrow \mathcal{X}$:

$$-\xi'(t) \in N_{\mathcal{C}(t)}(\xi(t)), \qquad \text{for a.e. } t \in [0, T], \tag{1.1}$$

$$\xi(0) = \xi^0, \tag{1.2}$$

where $\xi^0 \in \mathcal{C}(0)$ is a prescribed initial datum and

$$N_{\mathcal{K}}(x_0) := \{v \in \mathcal{X} \ : \ \langle v, x_0 - w \rangle \geq 0 \ \forall w \in \mathcal{K}\} \tag{1.3}$$

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is the exterior normal cone to a closed convex set $\mathcal{K} \subseteq \mathcal{X}$ at the point $x_0 \in \mathcal{K}$. Notice that it is implicitly assumed that

$$\xi(t) \in \mathcal{C}(t) \quad \forall t \in [0, T]. \tag{1.4}$$

Sweeping processes were introduced by J.J. Moreau in the fundamental paper [21] and originated a research which is still active: see, e.g., the monograph [20], the expository papers [17,29], and the references therein.

In the present paper we continue the analysis of [27], where we studied some continuity properties of the solution operator $\mathcal{C} \longmapsto \xi$ of the sweeping processes by setting it in the wider framework of *rate independent operators*, indeed problem (1.1)–(1.2) has the following property, called *rate independence*: if $\phi : [0, T] \longrightarrow [0, T]$ is an increasing surjective reparametrization of time and ξ is the solution associated to $\mathcal{C}(t)$, then $\xi(\phi(t))$ is the solution corresponding to $\mathcal{C}(\phi(t))$. Rate independent evolution problems are strictly connected to elasto-plasticity and hysteresis and have been deeply studied from the mathematical point of view in the monographs [12, 30,6,13,19]. The study of continuity properties with respect to various topologies has been recently performed also in, e.g., [17,5,16,31] and these properties are important since they ensure robustness of the model.

Here we address the sweeping process in the following formulation provided in [5]: a Banach space $\mathcal{Y}$, two functions $u : [0, T] \longrightarrow \mathcal{X}$, $r : [0, T] \longrightarrow \mathcal{Y}$, and a family of closed convex sets $\mathcal{Z}(r) \subseteq \mathcal{X}$ parametrized by $r \in \mathcal{Y}$ are given, and we have to find a function $\xi : [0, T] \longrightarrow \mathcal{X}$ such that

$$\langle u(t) - \xi(t) - z, \xi'(t)\rangle \geq 0, \qquad \text{for a.e. } t \in [0, T], \qquad \forall z \in \mathcal{Z}(r(t)), \tag{1.5}$$

$$u(0) - \xi(0) = x^0. \tag{1.6}$$

Again it is implicitly assumed that $u(t) - \xi(t) \in \mathcal{Z}(r(t))$ for all $t \in [0, T]$ (all the precise definitions, assumptions and formulations will be given in the next Sections 2 and 3).

Note that (1.5)–(1.6) is actually a reformulation of (1.1)–(1.2), indeed, as observed in [5], one can reduce (1.5)–(1.6) to (1.1)–(1.2) by setting $u(t) = 0$, $r(t) = t$, $x^0 = -\xi^0$, $\mathcal{C} = -\mathcal{Z}$; vice versa with the position $\mathcal{C}(t) = u(t) - \mathcal{Z}(r(t))$, $\xi^0 = u(0) - x^0$ one can reduce the first problem to the second. However formulation (1.5)–(1.6) introduces the new parameters $u(t)$, $r(t)$ that are relevant in applications, so that it is useful to study the properties of the sweeping process with respect to u and r. This analysis is performed in [5] where it is shown that the solution operator $\mathsf{S} : (u, r) \mapsto \xi$ of (1.5)–(1.6) is continuous with respect to the $W^{1,1}$-topology (or the strong BV topology, see (2.9)), i.e. if $u_n \to u$ in $W^{1,1}(0, T; \mathcal{X})$ and $r_n \to r$ in $W^{1,1}(0, T; \mathcal{Y})$, then $\mathsf{S}(u_n, r_n) \to \mathsf{S}(u, r)$ in $W^{1,1}(0, T; \mathcal{X})$. This property is essentially proved under some geometrical assumptions on $\mathcal{Z}(r)$ (cf. Assumption 3.1) which however turn out to be not so restrictive for applications.

In [21,15,16] the BV-generalization of (1.5)–(1.6) is considered: $\mathcal{Z}(r)$ is given as above, but u and r are with bounded variation, and one has to find a continuous function $\xi : [0, T] \longrightarrow \mathcal{X}$ of bounded variation such that (1.6) holds together with the condition

$$\int_0^T \langle u(t) - \xi(t) - z(t), \mathrm{d}\mathrm{D}\xi(t)\rangle \geq 0,$$
$$\forall z \in BV([0, T]; \mathcal{X}), \qquad z(t) \in \mathcal{Z}(r(t)) \quad \forall t \in [0, T], \tag{1.7}$$