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Multiplicity results for the scalar curvature equation

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Abstract

This paper is devoted to the study of positive radial solutions of the scalar curvature equation, i.e.

$$\Delta u(x) + K(|x|)u^{\sigma-1}(x) = 0$$

where $\sigma = \frac{2n}{n-2}$ and we assume that $K(|x|) = k(|x|^{\varepsilon})$ and $k(r) \in C^1$ is bounded and $\varepsilon > 0$ is small. It is known that we have at least a ground state with fast decay for each positive critical point of k for ε small enough. In fact if the critical point $k(r_0)$ is unique and it is a maximum we also have uniqueness; surprisingly we show that if $k(r_0)$ is a minimum we have an arbitrarily large number of ground states with fast decay. The results are obtained using Fowler transformation and developing a dynamical approach inspired by Melnikov theory. We emphasize that the presence of subharmonic solutions arising from zeroes of Melnikov functions has not appeared previously, as far as we are aware.

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1. Introduction

In this paper we study positive radial solutions for the following equation:

$$\Delta u(x) + K(|x|)u^{\sigma-1}(x) = 0 \tag{1.1}$$

where K is a bounded function, $x \in \mathbb{R}^n$, $\sigma = \frac{2n}{n-2}$ is the Sobolev critical exponent. Here and later for $u^{\sigma-1}$ we mean $(u_+)^{\sigma-1}$ where $u_+ = \max\{u, 0\}$; however, since we only deal with positive solutions the argument obviously works for the odd function $u^{\sigma-1} = u|u|^{\sigma-2}$, too. We focus on radial solutions, so we will in fact consider the following equation:

$$u''(r) + \frac{n-1}{r}u'(r) + K(r)u(r)^{\sigma-1} = 0 \tag{1.2}$$

where $r = |x|$. We want to apply singular perturbation methods so in fact we restrict our attention to K which is a perturbation of the following type:

S $K(r) = k(r^{\varepsilon})$, where $k(r) \in C^1$ is bounded, $\varepsilon > 0$ small enough.

Roughly speaking assuming **S** we require K to vary slowly, but it can also change sign. The main hypothesis will be that K admits a positive local minimum (but may assume negative values).

We recall some standard terminology. We say that a positive solution $u(r)$ of (1.2) is regular if $u(0) = u_0 > 0$ and $u'(0) = 0$, and that it is singular if $\lim_{r\to 0} u(r) = +\infty$. Moreover we say that $u(r)$ has fast decay (f.d.) if $\lim_{r\to+\infty} u(r)r^{n-2} = l$ and l is a positive constant and that it has slow decay (s.d.) if $l = +\infty$. In fact it can be shown, see e.g. [20,18], that "most" of singular and slow decay solutions are such that $u(r)r^{(n-2)/2}$ is positive and bounded respectively as $r \to 0$ and as $r \to +\infty$, when K is uniformly positive. We say that $u(r)$ is a Ground State (G.S.) if it is a regular solution which is positive for any $r \geq 0$, and consequently it is decaying as $r \to +\infty$; we say that $u(r)$ is a Singular Ground State (S.G.S.) if it is a singular solution which is positive for any $r > 0$.

Since we work directly on (1.2) our approach is suitable just for radials solutions. However if there is $R_0 > 0$ such that k is nonincreasing for $r < R_0$ and then nondecreasing for $r > R_0$, G.S. have to be radial; this fact was proved in [6] via moving plane techniques, and it is the setting we are mainly interested in. Furthermore positive solutions usually play a key role for the associated parabolic problems, see e.g. [23], and are one of the main tools to construct sub- and supersolutions.

In the last decades there has been a great interest in equations of type (1.1), see e.g. [3–9,12,11,14,18,20–22,24,26,27] for a definitely not-exhaustive review of the literature. The existence of positive solutions for (1.1) has application in several different areas, e.g. quantum mechanics and astrophysics, see e.g. [25]. Moreover equation (1.1) is also known as scalar curvature equation since the existence of G.S. with f.d. is equivalent to the existence of a metric in $\mathbb{R}^n$ conformally equivalent to the Euclidean metric and which has scalar curvature K, see e.g. [8,9] for more details.

We recall that when K is monotone increasing (1.1) is subcritical, so all the regular solutions become null with non-negative slope (crossing solutions), while when K is decreasing then (1.1) is supercritical, i.e. all the regular solutions are G.S. with s.d.: such a phenomenon was first