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A note on the strong maximum principle

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Abstract

We give a necessary and sufficient condition for the validity of the strong maximum principle in one space dimension.

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1. Introduction

The maximum principle is an important feature of second-order elliptic equations. Although its weak form is sufficient for many purposes (*e.g.*, see [12,19]), the *strong maximum principle*, which excludes the assumption of a nontrivial interior maximum, is often important. Its classical form can be stated as follows. Let $\Omega \subseteq \mathbb{R}^N$ be a connected, open set with smooth boundary $\partial\Omega$. Consider a uniformly elliptic operator of the form

$$\mathcal{L}u \equiv -\sum_{i,j=1}^{N} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i \frac{\partial u}{\partial x_i} + cu$$

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with coefficients $a_{ij} = a_{ji}$, $b_i, c \in C(\Omega) \cap L^\infty(\Omega)$ $(i, j = 1, \dots, N)$. Let $c \geq 0$ in Ω, and let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy the inequality $\mathcal{L}u \leq 0$ in Ω. Then the strong maximum principle asserts that, if there exists $x_0 \in \Omega$ such that $u(x_0) = \sup_\Omega u =: M \geq 0$, then $u \equiv M$ in Ω. It follows that, if $\mathcal{L}u \geq 0$ in Ω and there exists $x_0 \in \Omega$ such that $u(x_0) = \inf_\Omega u =: m \leq 0$, then $u \equiv m$ in Ω. In particular, by the strong maximum principle,

$$\textit{if } u \geq 0 \textit{ and } \mathcal{L}u \geq 0 \textit{ in } \Omega, \textit{ then either } u > 0 \textit{ in } \Omega, \textit{ or } u \equiv 0 \text{ in } \Omega\,.$$

Much effort has been made to extend the above result in two main directions: (i) addressing *degenerate* elliptic operators, for which the assumptions of uniform ellipticity, and/or regularity and boundedness of the coefficients a_{ij}, b_i, c are not satisfied; (ii) relaxing the assumptions concerning the boundary $\partial\Omega$. In connection with (i), weak or distributional formulations of the differential inequality $\mathcal{L}u \leq 0$ have been used.

Let us recall some results concerning degenerate elliptic operators. Let $a_{ij} \in C^2(\overline{\Omega})$, $b_i \in C^1(\overline{\Omega})$, $D^\alpha a_{ij} \in L^\infty(\Omega)$ for $|\alpha| \leq 2$, $D^\alpha b_i \in L^\infty(\Omega)$ for $|\alpha| \leq 1$; let $u \in C^2(\overline{\Omega})$ satisfy $\mathcal{L}u \leq 0$ in Ω. For any $x_0 \in \Omega$ such that $u(x_0) = \sup_\Omega u > 0$, consider the *propagation set* $\mathcal{P}(x_0) := \{x \in \Omega \,|\, u(x) = u(x_0)\}$. As proven in [24], $\mathcal{P}(x_0)$ contains the closure (in the relative topology) of the set $\mathcal{P}'(x_0)$ consisting of points, which can be joined to x_0 by a finite number of *subunitary* and/or *drift trajectories* (see [6,9,16,21] for the proof in particular cases; see also [1]). Remarkably, the set $\mathcal{P}'(x_0)$ coincides with the *support of the Markov process* corresponding to the operator $\mathcal{L}$ – namely, with the closure of the collection of all trajectories of a Markovian particle, starting at x_0, with generator $\mathcal{L}$ (see [23,24]). Therefore, only the *attainable* points of the boundary $\partial\Omega$ (see [10,11]) belong to $\mathcal{P}'(x_0)$.

A similar idea underlies the so-called *refined maximum principle* proven in [3], where the operator $\mathcal{L}$ is uniformly elliptic, but no smoothness of the boundary $\partial\Omega$ is assumed. Consider the subset of $E \subseteq \partial\Omega$ consisting of *attracting* boundary points, where the minimal positive solution U_0 of the *first exit time equation*

$$-\sum_{i,j=1}^{N} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i \frac{\partial u}{\partial x_i} = 1 \quad \text{in } \Omega \tag{1.1}$$

can be prolonged to zero (see [13,14]). It was proven in [3] that, if u is bounded from below in Ω, $u \geq 0$ in E and $\mathcal{L}u \geq 0$ in Ω, then $u \geq 0$ in Ω (in particular, in the usual situation the weak maximum principle is recovered).

In the above literature boundedness of the coefficients a_{ij}, b_i, c is always assumed. Concerning unbounded coefficients, while referring the reader to [8,15,20] and the references therein, let us focus on the following result. It is known that the strong maximum principle holds for the operator $\mathcal{L}u = -\Delta u + cu$, if $c \geq 0$ is measurable and $\sqrt{c} \in L^p_{loc}(\Omega)$ for some $p > N$ (see [22, Corollary 8.1]). More generally, the following holds (see [2], [7, Theorem 1], [17, Proposition 5.20]):

Theorem 1.1. *Let $\Omega \subseteq \mathbb{R}^N$ be connected and open. Let $c \in L^1(\Omega)$, $c \geq 0$* a.e. *in Ω, and let $u \in C(\Omega) \cap L^1(\Omega)$ be nonnegative in Ω. If $-\Delta u + cu \geq 0$ in $\mathcal{D}'(\Omega)$ and $u = 0$ in a set of positive Newtonian capacity, then $u \equiv 0$ in Ω.*

It was asked in [7] whether the condition $c \in L^1(\Omega)$ can be relaxed, *e.g.* by assuming $\sqrt{c} \in L^1_{loc}(\Omega)$. It is proven below that this is not the case (see Remark 2.2). This follows from the main