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Instability of internal equatorial water waves

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Abstract

In the following paper we present criteria for the hydrodynamical instability of internal equatorial water waves. We show, by way of the short-wavelength perturbation approach, that certain geophysical waves propagating above the equatorial thermocline are linearly unstable when the wave steepness exceeds a given threshold.

MSC: primary 76B15; secondary 76B47, 35B50, 26E05

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1. Introduction

In this paper we analyse the stability of some recently derived exact, explicit solutions to the geophysical governing equations in the equatorial β-plane using the short-wavelength perturbation method. The solutions we analyse prescribe steady, unidirectional, internal travelling waves which propagate above the thermocline, which is an interface separating two distinct vertical ocean layers of differing densities [9,11,17,33].

In [9] an explicit exact solution to the full geophysical governing equations was obtained which corresponds to the classical two-layer model describing oscillations of the thermocline in the equatorial region [17]. The solution in [9] is remarkable since, even in the setting where

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Coriolis effects are ignored, there are only a handful of explicit exact solutions to the full governing equations for water waves. Perhaps the most celebrated of these is Gerstner's wave [3,6,22,24], an exact solution that is explicit in the Lagrangian formulation [2,6]. Recently, quite a number of Gerstner-type exact and explicit solutions have been derived which model various physical and geophysical scenarios [3,4,7,9,11,25,28–31,36,38]. The formulation presented in [9] is quite unique in the sense that the fluid motion diminishes as one ascends from the thermocline towards the surface, as opposed to the Gerstner formulation for surface waves whereby motion decreases as one descends beneath the surface.

Recently, in [30], it was shown that the model describing internal waves propagating above the thermocline in [9] could be adapted to allow for a constant underlying current. This idea extends back to [37] when Mollo-Christensen introduced a current-like term into Gerstner's solution for gravity waves in order to describe billows between two fluids, and it was recently employed in the geophysical setting in [25]. The solutions presented in [9,30] represent explicit examples of exact equatorial waves whereby the fluid motion dies out at great depth, and we note that the recent papers [8,10,27] rigorously established the existence of exact equatorial surface waves which admit an underlying vorticity distribution.

In this paper we apply the short-wavelength perturbation method to derive instability criteria for the internal wave solutions which were obtained in [9,30]. These criteria are stated in Propositions 4.1 and 4.2 below. From a mathematical viewpoint, establishing the hydrodynamical stability or instability of a flow is difficult, given that the fully nonlinear governing equations for fluid motion are highly intractable [15,16,19,20]. Physically, the question of hydrodynamic stability is important since, for instance, unstable flows cannot be observed in practice since they are rapidly destroyed by any minor perturbations or disturbances. For certain solutions which have an explicit Lagrangian formulation, it transpires that the short-wavelength perturbation method of instability analysis, which was independently developed by the authors of [1,18,35], has a remarkably elegant formulation and application. This was first established for Gerstner's solution to the gravity water wave problem in [34], and for geophysical flows in [12]. Recent work [21,26,32] has applied the instability analysis to a variety of contexts. The current paper is, to the best of our knowledge, the first application of this approach to internal waves propagating above the equatorial thermocline.

2. Governing equations and model

We take the earth to be a perfect sphere of radius $R = 6378$ km with constant rotational speed of $\Omega = 73 \cdot 10^{-6}$ rad/s, and $g = 9.8\ \mathrm{m\,s^{-2}}$ is the gravitational acceleration at the surface of the earth. In a reference frame with the origin located at a point on earth's surface and rotating with the earth, we take the x-axis to be the longitudinal direction (horizontally due east), the y-axis to be the latitudinal direction (horizontally due north) and the z-axis to be vertically upwards. The governing equations for geophysical ocean waves [14] are given by

$$u_t + uu_x + vu_y + wu_z + 2\Omega w \cos\phi - 2\Omega v \sin\phi = -\frac{1}{\rho} P_x, \tag{2.1a}$$

$$v_t + uv_x + vv_y + wv_z + 2\Omega u \sin\phi = -\frac{1}{\rho} P_y, \tag{2.1b}$$

$$w_t + uw_x + vw_y + ww_z - 2\Omega u \cos\phi = -\frac{1}{\rho} P_z - g, \tag{2.1c}$$