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Eventual smoothness and asymptotics in a three-dimensional chemotaxis system with logistic source

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Abstract

We prove existence of global weak solutions to the chemotaxis system

$$u_t = \Delta u - \nabla \cdot (u\nabla v) + \kappa u - \mu u^2$$
$$v_t = \Delta v - v + u$$

under homogeneous Neumann boundary conditions in a smooth bounded convex domain $\Omega \subset \mathbb{R}^n$, for arbitrarily small values of $\mu > 0$.

Additionally, we show that in the three-dimensional setting, after some time, these solutions become classical solutions, provided that κ is not too large. In this case, we also consider their large-time behaviour: We prove decay if $\kappa \leq 0$ and the existence of an absorbing set if $\kappa > 0$ is sufficiently small.

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1. Introduction

Starting from the pioneering work of Keller and Segel [9], an extensive mathematical literature has grown on the Keller–Segel model and its variants, mathematical models describing chemotaxis, that is the tendency of (micro-)organisms to adapt the direction of their (otherwise random) movement toward increasing concentrations of a signalling substance. For a survey see [6] or [7,8].

If biological phenomena where chemotaxis plays a role are modelled not only on small timescales, often the growth of the population, whose density we will denote by u, must be taken into account. A prototypical choice to accomplish this is the addition of logistic growth terms $+\kappa u - \mu u^2$ in the evolution equation for u. Here $+\kappa u$, with $\kappa \in \mathbb{R}$ being the difference between birth rate and death rate of the population, is used to describe population growth, and the term $-\mu u^2$ models additional overcrowding effects. Negative values of κ can be used to include effects like spontaneous degradation into the model (e.g. in the case of a starving population) that – in contrast to the effects modelled by the quadratic term – take place also in regions with small population density. Unfortunately, it is unclear whether global classical solutions to the chemotaxis-system

$$\begin{aligned}
u_t &= \Delta u - \nabla \cdot (u \nabla v) + \kappa u - \mu u^2 \\
v_t &= \Delta v - v + u \\
\partial_\nu u|_{\partial\Omega} &= \partial_\nu v|_{\partial\Omega} = 0 \\
u(\cdot, 0) &= u_0, \qquad v(\cdot, 0) = v_0,
\end{aligned} \tag{1}$$

where u_0, v_0 are given functions, exist in the smooth, bounded domain $\Omega \subset \mathbb{R}^n$ if $n \geq 3$ and $\mu > 0$ is small.

The parabolic–elliptic simplification (where v_t is replaced by 0) of (1) has been considered in [23], where – besides some study of asymptotic behaviour – it is shown that weak solutions exist for arbitrary $\mu > 0$ and that they are smooth and globally classical if $\mu > \frac{n-2}{n}$. In [24] the existence of (very) weak solutions is proven under more general conditions. Under additional assumptions, also the existence of a bounded absorbing set in $L^\infty(\Omega)$ is shown.

Turning to the parabolic–parabolic system, important findings are given in [26], which assert existence and uniqueness of global, smooth, bounded solutions to (1) under the condition that μ be large enough.

Additional results on existence of global solutions or even of an exponential attractor have been given in the two-dimensional case (see e.g. [16,17]). In this case, global solutions exist for arbitrary $\mu > 0$.

Not only the restriction to dimension 2, but also the inclusion of some kind of saturation effect in the chemotactic sensitivity [2], sublinear dependence of the chemotactic sensitivity on u [3] or even changing the second equation into one that models the consumption of the chemoattractant (as done in [20,22] for $\kappa = \mu = 0$) can make it possible to derive the global existence of solutions. The same can be accomplished by replacement of the secretion term $+u$ in the second equation of (1) by $+\frac{u}{(1+u)^{1-\beta}}$ with some $0 < \beta < \frac{9}{10}$, which enables the authors of [14] to show the existence of attractors in the corresponding dynamical system.