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On differentiability of eigenvalues of second order elliptic operators on non-smooth domains

Julian Haddad, Marcos Montenegro *

Departamento de Matemática, Universidade Federal de Minas Gerais, Caixa Postal 702, 30123-970, Belo Horizonte, MG, Brazil

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Abstract

The regularity of eigenvalues of elliptic operators upon deformations of a given bounded domain is a classical problem in elliptic PDEs which has been focused by many authors. We establish a theorem on C^r dependence of algebraically simple eigenvalues and eigenfunctions with respect to perturbations of C^1 class of non-smooth domains and of C^r class of coefficients of elliptic operators. Moreover, we also compute the first variation of these eigenvalues in relation to both parameters for non-smooth domains. As a byproduct, we extend Hadamard's formula to second order elliptic operators for domains of C^2 class and other non-smooth ones.

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* Corresponding author.
E-mail addresses: julianhaddad@gmail.com (J. Haddad), montene@mat.ufmg.br (M. Montenegro).

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1. Introduction

This work proposes the study on the dependence of eigenvalues and eigenfunctions of the following linear boundary value problem

$$\begin{cases} -\mathcal{L}_{\Omega,\mathcal{C}}\, u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

with respect to the domain Ω and coefficients of the operator $\mathcal{L}_{\Omega,\mathcal{C}}$.

Here, Ω denotes a general bounded domain in $\mathbb{R}^n$, $n \geq 1$, $\lambda \in \mathbb{R}$ and $\mathcal{L}_{\Omega,\mathcal{C}}$ stands for a second order uniformly elliptic linear operator of the form

$$\mathcal{L}_{\Omega,\mathcal{C}} := \sum_{i,j=1}^{n} \partial_i (a_{ij}(x)\partial_j) - \sum_{i=1}^{n} b_i(x)\partial_i - c(x)$$

with coefficients a_{ij}, b_i and c in $C^r(\overline{U})$, where $r \geq 1$ and U is a fixed bounded open subset of $\mathbb{R}^n$ such that $\overline{\Omega} \subset U$. The symbol $\mathcal{C}$ represents the N-tuple of coefficients $\mathcal{C} = (a_{ij}, b_i, c)$ with $N = n^2 + n + 1$. Note that any second order linear differential operator with coefficients in $C^1(\overline{U})$ can always be placed into the above format.

This has been a classical theme in the theory of elliptic PDEs presumably since the famous Rayleigh's book [45] published in 1877 (first edition). There is a vast literature concerning this problem in the 20th century: Hadamard [22] in 1908, Courant and Hilbert [13] in the German edition of 1937, Polya and Szëgo [43] in 1951, Garabedian and Schiffer [19,20] in 1952–1953, Polya and Schiffer [42] in 1953, Schiffer [46] in 1954 and thereafter [2,18,3,27,39,36–38,40,47,16,44,14,17,21]. For more recent advances, we cite the works [29,11,15,35,6,26,25,30,31,4,8,9,33,10,34,32]. We also mention three interesting works on generic properties of eigenvalues and eigenfunctions due to Uhlenbeck [48,49] and Pereira [41] which are closely related to the issue of this paper. Although this list is far from complete, it shows how certain eigenvalues problems for elliptic operators are attractive.

Part of these works have dedicated special attention to the study on regularity of eigenvalues of elliptic operators with respect to perturbations of a given domain. The reason of the great interest on this question lies at its applicability. For instance, we mention two problems: to know the nature of critical domains for the eigenvalue functional subject to domains with fixed volume and also the expansion of eigenvalues around a given domain, see [43,19,42,46,26]. The pure discussion on regularity of eigenvalues upon smooth deformations of the domain also plays a key role in global dynamic of certain types of parabolic PDEs, namely in the study of stability properties of equilibria. More specifically, an underlying question is to know whether eigenvalues corresponding to one-parameter perturbations Ω_ε of a fixed domain Ω are differentiable with respect to ε. Results on spectral stability can be found in [13,28,12,24,47,7] and references therein. See also Hale's survey [23] on this topic.

Regarding perturbations which are described as embedding maps, the first result on differentiability of eigenvalues with respect to the domain can be found in the book by Rayleigh [45]. A general technique was proposed by Hadamard [22], who considered smooth perturbations of a particular domain with smooth boundary. For more general smooth domains, the continuity of eigenvalues was studied by Courant and Hilbert [13] (Theorem 11, p. 423), Babuška and