Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 258 (2015) 362–378

Journal of Differential Equations

www.elsevier.com/locate/jde

On the (non) existence of viscosity solutions of multi-time Hamilton–Jacobi equations

A. Davini [a], M. Zavidovique [b,*,1]

[a] Dip. di Matematica, Sapienza Università di Roma, P.le Aldo Moro 2, 00185 Roma, Italy
[b] IMJ-PRG (projet Analyse Algébrique), UPMC, 4, place Jussieu, Case 247, 75252 Paris Cédex 5, France

Received 21 March 2014; revised 13 August 2014

Available online 11 October 2014

Abstract

We prove that the multi-time Hamilton–Jacobi equation in general cannot be solved in the viscosity sense, in the non-convex setting, even when the Hamiltonians are in involution.

MSC: 35F21; 49L25; 37J50

Keywords: Commuting Hamiltonians; Viscosity solutions; Symplectic geometry

0. Introduction

There are two important Hamilton–Jacobi equations associated with a continuous function H defined on the cotangent bundle of a smooth Riemannian manifold M: the stationary equation, that is a nonlinear PDE of the kind

$$H(x, d_x u) = c \quad \text{in } M,$$

where c is a real constant and the unknown u is a real function defined on M; and the evolutionary equation, which is a time-dependent equation of the form

* Corresponding author.
E-mail addresses: davini@mat.uniroma1.it (A. Davini), zavidovique@math.jussieu.fr (M. Zavidovique).
[1] The second author is financed by ANR-12-BLAN-WKBHJ.

http://dx.doi.org/10.1016/j.jde.2014.09.015

$$\begin{cases} \partial_t u + H(x, d_x u) = 0 & \text{in } (0, +\infty) \times M \\ u(0, x) = u_0(x), & \text{in } M, \end{cases} \tag{0.1}$$

where $u_0 : M \to \mathbb{R}$ is an initial datum, and the unknown u is a real function of variables $(t, x) \in [0, +\infty) \times M$.

The first problem that naturally arises is that of finding a good notion of solution. It is in fact well known that such equations do not usually admit classical solutions. For instance, the method of characteristics provides, for smooth initial data and under general assumptions on H, classical solutions for (0.1) only for small times, until shocks between characteristics occur.

A first idea is to look for Lipschitz functions which solve the equation almost everywhere. However, such a notion of solution is inadequate since it lacks of good uniqueness and stability properties, see for instance [2]. That is why two better notions of weak solutions were introduced.

The first one, introduced by Crandall and Lions in 1983 [11], is that of *viscosity solution* and it applies to a wide range of first and second order nonlinear PDE. It is in some way reminiscent of distributions, in the sense that it uses test functions to drop derivatives, and permits to name solutions functions that are just continuous, regardless of their regularity. This notion has revealed to be extremely powerful and flexible, and it has been generalized since then in many different directions.

The second one, more geometric in nature, is termed *variational solution* and was introduced by Chaperon, Sikorav and Viterbo (see [10] or [9] and references therein) and is more specific of evolutionary equations of the kind of (0.1). It follows the idea of the method of characteristics, taking the graph of the differential of the initial condition and pushing it with the Hamilton flow. When the image is no longer a graph, a natural way to cut the obtained Lagrangian manifold, known as the graph selector, allows to reconstruct a function which is the desired variational solution.

Note that, as can be expected, both those notions often yield almost everywhere solutions (meaning Lipschitz functions satisfying the Hamilton–Jacobi equation almost everywhere). In the case of a Hamiltonian $H(x, p)$ which is convex in the momentum variable p, it was proved by Zhukovskaya (see [23] and also [6,20]) that both notions provide the same solution. However, they may differ in the general case. Indeed, variational solutions do not necessarily verify the semi-group (or Markov) property. There is a simple way (proposed by Chaperon) to force variational solutions to verify this property and Wei [20] recently established that the obtained functions are the viscosity solutions (see also the work of Roos [19] on this matter).

In this paper we will focus on the solvability of the following system of uncoupled Hamilton–Jacobi equations:

$$\begin{cases} \partial_t u + H(x, d_x u) = 0 & \text{in } (0, +\infty) \times (0, +\infty) \times M \\ \partial_s u + G(x, d_x u) = 0 & \text{in } (0, +\infty) \times (0, +\infty) \times M \\ u(0, 0, x) = u_0(x) & \text{on } M. \end{cases}$$

Such a system is known as *multi-time Hamilton–Jacobi equation* and has its roots in economics. There is now a consequent literature concerning this problem, where either one or the other notion of solution has been considered. The issue about existence and uniqueness of viscosity solutions of the multi-time equation was first addressed by Lions and Rochet [17] in the case of convex and coercive Hamiltonians on $\mathbb{R}^n \times \mathbb{R}^n$ only depending on the momentum variable. This work was subsequently generalized in [3] to Hamiltonians depending on both variables of $\mathbb{R}^n \times \mathbb{R}^n$, but still convex and coercive in the momentum. As a counterpart, a commutation hypothesis is