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Existence, uniqueness and regularity for a class of semilinear stochastic Volterra equations with multiplicative noise

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Abstract

We consider a class of semilinear Volterra type stochastic evolution equation driven by multiplicative Gaussian noise. The memory kernel, not necessarily analytic, is such that the deterministic linear equation exhibits a parabolic character. Under appropriate Lipschitz-type and linear growth assumptions on the nonlinear terms we show that the unique mild solution is mean-p Hölder continuous with values in an appropriate Sobolev space depending on the kernel and the data. In particular, we obtain pathwise space–time (Sobolev–Hölder) regularity of the solution together with a maximal type bound on the spatial Sobolev norm. As one of the main technical tools we establish a smoothing property of the derivative of the deterministic evolution operator family.

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1. Introduction

We consider a stochastic evolution equation of Volterra type driven by multiplicative Gaussian noise given in the Itô form

$$\begin{gathered} \mathrm{d}u + \left(A \int_0^t b(t-s)u(s)\mathrm{d}s \right)\mathrm{d}t = F(u)\mathrm{d}t + G(u)\mathrm{d}W, \quad t > 0, \\ u(0) = u_0. \end{gathered} \tag{1.1}$$

The process $\{u(t)\}_{t\in[0,T]}$, defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ with a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$, takes values in a separable Hilbert space H with inner product $(\cdot,\cdot)$ and induced norm $\|\cdot\|$. The process W is a nuclear Q-Wiener process with respect to the filtration with values in some separable Hilbert space U. The operator $A : \mathcal{D}(A) \subset H \to H$ is assumed to be linear, unbounded, self-adjoint and positive definite. The main example we have in mind for A is the Dirichlet Laplacian on $H = L^2(\mathcal{O})$, where $\mathcal{O} \subset \mathbb{R}^d$ is a spatial domain with smooth boundary. Throughout the paper the kernel b is kept as general as possible but so that the deterministic, linear, homogeneous version of (1.1) exhibits a parabolic character. In particular, we assume that the Laplace transform $\hat{b}$ of b maps the right half-plane into a sector around the real axis with central angle less than π, and $\hat{b}$ satisfies some regularity and growth conditions, see Assumption 2.4 and Remark 2.5 for the precise conditions on b. One of the important kernels that satisfies this assumption is the tempered Riesz kernel $b(t) = \frac{1}{\Gamma(\rho-1)} t^{\rho-2} e^{-\eta t}$, where $1 < \rho < 2$ and $\eta \geq 0$. The main goal of the paper is to extend the results of [8] and [14], where SPDEs without a memory term are considered, to the solution of (1.1) under analogous, appropriate Lipschitz and linear growth assumptions on f and G, see Assumption 2.8. That is, in Theorem 3.3, we prove existence, uniqueness, and mean-p Hölder regularity in time with values in fractional order spaces $\dot{H}^\beta$ associated with the fractional powers of A (see, Section 2.2). Then corresponding pathwise regularity results immediately follow, see Corollary 3.6.

There are several approaches to stochastic partial differential equations and then of course to Volterra type stochastic partial differential equations as well. Firstly, one chooses a framework for infinite dimensional stochastic integration. One possibility, as in the present paper, is to choose abstract stochastic integration theory in Hilbert spaces, such as in [6] and [17]. Then, one has the option to consider a semigroup framework from [6] with a suitable state space that incorporates the history of the process as, for example, in [1,2]. In the latter papers existence and uniqueness are established for a class of semilinear Volterra type SPDEs with multiplicative noise under partly more general and partly more restrictive assumptions than in this paper and space–time regularity is not investigated.

The other option for defining solutions to (1.1), which we also choose to follow, is the resolvent family approach of Prüss [19] based on the Laplace transform. This has been mainly used to study linear equations with additive noise, see [4,9–11,20], with the exception of [3] and [12] where a semilinear equation with additive noise and, respectively, multiplicative noise is considered. All these papers mainly concern existence and uniqueness, and not so much with regularity, apart from some limited analysis in the linear additive case [4,20].

Finally, an other possibility is to use Krylov's approach for stochastic integration in case H is specifically $L^2(\mathcal{O})$ space (more generally $L^p(\mathcal{O})$) where stochastic integrals are taken pointwise. This approach is taken, for example, in [5] where a semilinear Volterra type equation is considered with the specific kernel $b(t) = \frac{1}{\Gamma(\rho-1)} t^{\rho-2}$. There the authors obtain regularity results (but