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Cyclicity of a fake saddle inside the quadratic vector fields

P. De Maesschalck [a], S. Rebollo-Perdomo [b], J. Torregrosa [c,*]

[a] *Hasselt University, Martelarenlaan 42, B-3500 Hasselt, Belgium*
[b] *Centre de Recerca Matemàtica, 08193 Bellaterra, Barcelona, Spain*
[c] *Departament de Matemàtiques, Universitat Autònoma de Barcelona, Edifici C, 08193 Bellaterra, Barcelona, Spain*

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Abstract

This paper concerns the study of small-amplitude limit cycles that appear in the phase portrait near an unfolded fake saddle singularity. This degenerate singularity is also known as an impassable grain. The canonical form of the unperturbed vector field is like a degenerate flow box. Near the singularity, the phase portrait consists of parallel fibers, all but one of which have no singular points, and at the singular fiber, there is one node. We demonstrate different techniques in order to show that the cyclicity is bigger than or equal to two when the canonical form is quadratic.

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Keywords: Cyclicity; Fake saddle; Impassable grain; Limit cycle; Bifurcation; Singularity unfolding

* Corresponding author.
E-mail addresses: peter.demaesschalck@uhasselt.be (P. De Maesschalck), srebollo@crm.cat (S. Rebollo-Perdomo), torre@mat.uab.cat (J. Torregrosa).

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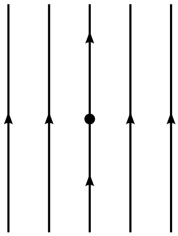

Fig. 1. Phase portrait of X_0.

1. Introduction

This paper concerns the study of small-amplitude limit cycles that appear in the phase portrait near an unfolded degenerate singularity. More specifically, we assume that the unperturbed vector field can be put in a form that is like a degenerate flow box: near the singularity, the phase portrait consists of parallel fibers, all but one of which have no singular points, and the singular fiber has a semi-stable equilibrium point. This singularity is known as a fake saddle or an impassable grain, see [24]. In fact, it is a singularity with exactly two saddle sectors.

Though the paper deals with more general vector fields, to present the ideas, consider the following typical model:

$$X_0\colon \left\{\dot{x} = 0,\ \dot{y} = x^2 + y^2\right\},$$

whose local phase portrait is shown in Fig. 1.

In any unfolding, the orbits $\{x = \text{const}\}$ away from the origin will smoothly be perturbed in a trivial way. Close to the origin, more complicated phenomena may occur: we show the presence of Hopf bifurcations, Bogdanov–Takens bifurcations, slow–fast (canard) behavior, homoclinic and heteroclinic orbits. All the above phenomena are well-known mechanisms near which limit cycles can be born, and in fact the study of periodic orbits near the degenerate point is the principle goal of this paper. We use the aforementioned mechanisms to show the presence of up to two small amplitude limit cycles, and provide evidence that by using these mechanisms this is the best cyclicity result one can get.

Determining an upper bound for the number of limit cycles turned out to be too difficult, as it was revealed that a multi-parameter global study of phase portraits was needed, going far beyond the traditional perturbative methods to create limit cycles.

In a study of unfoldings of a vector field like X_0, it is best to make a homogeneous (family) blow-up of the perturbed family of vector fields, thereby focusing on the behavior at the blow-up locus. The behavior at the blow-up locus has been shown to be mostly determined by perturbation terms of degree two and lower. *We will therefore focus our attention on perturbations of at most degree two.* Though this restriction immediately shows a relation between the Hilbert 16th problem in degree 2, the study of the singularity at X_0 has in fact no contribution in the degree-2 program outlined by Dumortier, Roussarie and Rousseau [14]. In that program, homogeneous vector fields could be avoided using rescalings. Setting this point aside, the study of the cyclicity of X_0 at the origin has a relevance by itself.

In Section 2 we consider forms for the unperturbed system under some additional generic and geometric constraints, and present a canonical form that depends on two parameters (A, B). Next, we present a reduction to canonical form of *the unfolding* of the degenerate singular point.