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Uniqueness theorems for differential pencils with eigenparameter boundary conditions and transmission conditions

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ABSTRACT

Inverse spectral problems are considered for differential pencils with boundary conditions depending polynomially on the spectral parameter and with a finite number of transmission conditions. We give formulations of the associated inverse problems such as Titchmarsh–Weyl theorem, Hochstadt–Lieberman theorem and Mochizuki–Trooshin theorem, and prove corresponding uniqueness theorems. The obtained results are generalizations of the similar results for the classical Sturm–Liouville operator on a finite interval.

1. Introduction

In this article, we investigate the problem $L(p,q,P_0,P_1,R_0,R_1,d_i,a_i,b_i)$ of the form

$$ly(x) := -y''(x) + \left[2\lambda p(x) + q(x)\right]y(x) = \lambda^2 y(x), \quad x \in \bigcup_{i=0}^{m-1}(d_i, d_{i+1}), \tag{1.1}$$

where the real-valued functions $p, q \in L^1(0,\pi)$ and λ is the spectral parameter, and with boundary conditions

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$$U(y) := P_1(\lambda)y'(0) - P_0(\lambda)y(0) = 0, \qquad V(y) := R_1(\lambda)y'(\pi) + R_0(\lambda)y(\pi) = 0, \tag{1.2}$$

where $P_1(\lambda)$, $P_0(\lambda)$, $R_1(\lambda)$, $R_0(\lambda)$ are polynomials with respect to λ, and with transmission conditions

$$U_i(y) := y(d_i + 0) - a_i y(d_i - 0) = 0 \tag{1.3}$$

and

$$V_i(y) := y'(d_i + 0) - a_i^{-1} y'(d_i - 0) - b_i y(d_i - 0) = 0, \tag{1.4}$$

where a_i, b_i, d_i, $i = 1, 2, \ldots, m-1$ (with $m \geqslant 1$) are real numbers, satisfying $a_i > 0$ and $d_0 = 0 < d_1 < d_2 < \cdots < d_{m-1} < d_m = \pi$.

The values of the parameter λ for which $L(p, q, P_0, P_1, R_0, R_1, d_i, a_i, b_i)$ has non-zero solutions are called eigenvalues, and the corresponding non-trivial solutions are called eigenfunctions. The set of eigenvalues is called the spectrum of the problem $L(p, q, P_0, P_1, R_0, R_1, d_i, a_i, b_i)$.

Differential equations with the spectral parameter and transmission conditions arise in various problems of mathematics as well as in applications. Detailed studies on spectral problems for ordinary differential operators depending on the parameter and/or with transmission conditions can be found in various publications, see e.g. [1,2,4,7,9,10,13,16,17,21,22,27–30,32,33,35,39], where further references and links to applications can be found.

The inverse spectral problem for the problem $L(p, q, P_0, P_1, R_0, R_1, d_i, a_i, b_i)$ is studied in this paper. Inverse problems of spectral analysis under consideration consist in recovering operators from their spectral characteristics. As the main spectral characteristic we introduce the Weyl function which is a generalization of Weyl's $m(\lambda)$-function for the Sturm–Liouville operator. We note that for the classical Sturm–Liouville operators the inverse problem has been studied fairly completely (see e.g., [8,18,19,25,40]), and for the differential pencils the inverse problem also was studied (see e.g., [5,6,11,36,38,41]). In particular the non-linear appearance of the spectral parameter in the boundary conditions and a finite number of transmission conditions leads in connection with inverse spectral problems of differential pencils to additional difficulties which could, up to now, not have been solved.

One of the earliest results on half inverse problems for Sturm–Liouville operators is due to Hochstadt and Lieberman [14]. They consider the problem

$$-y''(x) + q(x)y(x) = \lambda^2 y(x) \quad \text{on } (0, \pi),\ q \in L^1(0, \pi),$$
$$y(0)\cos\alpha - y'(0)\sin\alpha = 0, \qquad y(\pi)\cos\beta + y'(\pi)\sin\beta = 0, \quad \alpha \in [0, \pi),\ \beta \in [0, \pi),$$

and prove that, for fixed β, if the complex valued function $q(x)$ is known on $(\pi/2, \pi)$, then a single spectrum $\{\lambda_n\}$ suffices to determine $q(x)$ on $(0, \pi/2]$ and α.

Hald [13] generalized a theorem in [14]. Gesztesy and Simon [12] found new uniqueness results with partial information on the spectrum for Sturm–Liouville operators with scalar and matrix coefficients, respectively. They showed that more information on the potential can compensate for less information about the spectrum. Martinyuk and Pivovarchik [20] proposed a new method for reconstructing the potential on half the interval. Sakhnovich [26] studied the existence of solutions of half inverse problems. Singular potentials were studied by Hryniv and Mykytyuk [15]. Buterin studied half inverse problem for differential pencils with the spectral parameter in boundary conditions [3]. Trooshin and Yamamoto [31] obtained Hochstadt–Lieberman type theorems for nonsymmetric first order systems. For quadratic pencils of Sturm–Liouville operators without the spectral parameter and transmission conditions Yang and Zettl [37] proved that if $p(x)$ and $q(x)$ are known on half of the domain interval, then one spectrum suffices to determine them uniquely on the other half. These references are certainly not intended to be comprehensive but are given to indicate the wide interest in and variety of half inverse type problems.

One of the earliest results on the inverse problem for interior spectral data for Sturm–Liouville operators is due to Mochizuki and Trooshin [23]. Inverse problem for interior spectral data of the