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Regularity of solutions to the liquid crystal flows with rough initial data

Junyu Lin

Department of Mathematics, South China University of Technology, Guangzhou, Guangdong 510640, China

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ABSTRACT

In this paper, we are concerned with the regularity of solutions to the liquid crystal flows with rough initial data in R^n. We prove that the solution constructed by Wang (2011) in [23] has higher regularity. Moreover we obtain a decay estimate in time for any space derivative.

1. Introduction

In this paper, we will study the following equation modeling the hydrodynamic flow of nematic liquid crystals, which has been proposed and investigated by Lin and Liu [15,16] and [17]:

$$v_t + \nabla \cdot (v \otimes v) - \Delta v + \nabla \Pi = -\nabla \cdot (\nabla d \odot \nabla d) \quad \text{in } R^n \times (0, \infty), \tag{1.1}$$

$$\nabla \cdot v = 0 \quad \text{in } R^n \times (0, \infty), \tag{1.2}$$

$$d_t + v \cdot \nabla d = \Delta d + |\nabla d|^2 d \quad \text{in } R^n \times (0, \infty), \tag{1.3}$$

$$v(x, 0) = v_0(x), \qquad d(x, 0) = d_0(x), \qquad \left|d_0(x)\right| = 1 \quad \text{in } R^n, \tag{1.4}$$

where $v = (v_1, \ldots, v_n) : R^n \times (0, \infty) \to R^n$ represents the fluid velocity field, $d = (d_1, d_2, d_3) : R^n \times (0, \infty) \to S^2$ is the director field of the nematic liquid crystals and $\Pi : R^n \times (0, \infty) \to R$ is the pressure function. $v \otimes v$ denotes a tensor product whose (i, j)-th entry is given by $v_i v_j$ for $1 \leqslant i, j \leqslant n$. $\nabla d \odot \nabla d$ denotes the $n \times n$ matrix whose (i, j)-th entry is given by $\nabla_i d \cdot \nabla_j d$ for $1 \leqslant i, j \leqslant n$.

E-mail address: linjiny79@gmail.com.

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The system (1.1)–(1.4) is a simplified version of the Ericksen–Leslie model [4,5,13], but still retains most of the interesting mathematical properties of the original Ericksen–Leslie model for the hydrodynamics of nematic liquid crystals. See [15–17] for more discussions on the relations of the two models. In principle, the system (1.1)–(1.4) is a macroscopic continuum description of the time evolutions of these materials under the influence of both the flow field v, and the macroscopic description of the microscopic orientational configurations d, which can be derived from the averaging/coarse graining of the directions of rod-like liquid crystal molecules. Mathematically, (1.1)–(1.4) is a strongly coupled system between the incompressible Navier–Stokes equation and the transported heat flow of harmonic maps.

Lin, Lin and Wang [19] have proved that there is a global Leray–Hopf type weak solution to (1.1)–(1.4) with boundary condition in 2D, which is smooth away from at most finitely many singular times. See also Hong [8] for related work. Authors in [27] have established a global existence and regularity of weak solution for (1.1)–(1.4) in 2D with the large initial velocity and have proved the uniqueness of weak solution by using the Littlewood–Paley analysis.

For higher dimensions, Lin and Wang [20] have established the uniqueness for the class of Leray–Hopf type weak solutions derived in [19] and uniqueness of weak solution $(v, d) \in C([0, T]; L^n(R^n)) \times C([0, T]; W^{1,n}(R^n, S^2))$ to (1.1)–(1.4) in n-dimensions for $n \geqslant 3$ with initial data (v_0, d_0) satisfying $v_0, \nabla d_0 \in L^n(R^n)$. Lin and Ding [14] have established that for $v_0, \nabla d_0 \in L^n(R^n)$, if $(\|v_0\|_{L^n(R^n)} + \|\nabla d_0\|_{L^n(R^n)})$ is small enough, then (1.1)–(1.4) has a unique global solution (v, d) with $v, \nabla d \in C([0, +\infty); L^n(R^n))$.

For rough initial data, Wang [23] have proved that if $(\|v_0\|_{BMO^{-1}(R^n)} + [d_0]_{BMO(R^n)})$ is small enough, there exists a mild solution (v, d) of (1.1)–(1.4) in $Z \times X$. Recently, Lin in [12] established a uniqueness criterion of liquid crystal flows in higher dimensions. He proved that the mild solution (v, d) of liquid crystal flows is unique under some class data in $vmo^{-1}(R^n) \times vmo(R^n)$, where the spaces BMO, BMO^{-1}, vmo and vmo^{-1} will be explained later.

As mentioned above, (1.1)–(1.3) is a coupled system between the incompressible Navier–Stokes equation and the transported heat flow of harmonic maps. It is well known that the weak solutions exist globally and are partial regular for both incompressible Navier–Stokes equation [1,7,11] and heat flow of harmonic maps [2,3,22,24]. Though there are some results on the incompressible flows of liquid crystals (see [25,26]) in dimensions three, but it is still an open problem for the global existence and regularity of weak solution (1.1)–(1.4) in dimensions $n \geqslant 3$ (see [10]). In this paper, we will focus on the regularity of solutions to (1.1)–(1.4) with rough initial data.

To state our result, we recall some definitions and notations. Let

$$K(x, t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$$

denote the heat kernel and $e^{t\Delta} = K(\cdot, t)*$ denote the heat semigroup.

Definition 1.1. Let $f(x)$ be a measurable function in R^n. For $0 < T \leqslant +\infty$, define

$$\|f\|_{BMO_T^{-1}} := \sup_{x \in R^n,\, 0<r^2<T} \left(r^{-n} \int_{B_r(x)} \int_0^{r^2} \left| e^{t\Delta} f(y) \right|^2 dt\, dy \right)^{\frac{1}{2}},$$

$$[f]_{BMO_T} := \sup_{x \in R^n, 0<r^2<T} \left(r^{-n} \int_{B_r(x)} \int_0^{r^2} \left| \nabla e^{t\Delta} f(y) \right|^2 dt\, dy \right)^{\frac{1}{2}},$$

where $B_r(x)$ is a ball of radius r and centered at x. Denote

$$BMO_T^{-1}(R^n) := \left\{ f(x) \in S'(R^n) \;\middle|\; \|f\|_{BMO_T^{-1}} < \infty \right\};$$