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Convergence of scalar-flat metrics on manifolds with boundary under a Yamabe-type flow

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Abstract

We study a conformal flow for compact Riemannian manifolds of dimension greater than two with boundary. Convergence to a scalar-flat metric with constant mean curvature on the boundary is established in dimensions up to seven, and in any dimensions if the manifold is spin or if it satisfies a generic condition.

MSC: 35J65; 53C25

Keywords: Yamabe flow; Manifold with boundary; Conformal metric; Scalar curvature; Mean curvature

1. Introduction

Let M^n be a closed manifold with dimension $n \geq 3$. In order to solve the Yamabe problem (see [39]), R. Hamilton introduced the Yamabe flow, which evolves Riemannian metrics on M according to the equation

$$\frac{\partial}{\partial t} g(t) = -(R_{g(t)} - \overline{R}_{g(t)})g(t)\,,$$

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where R_g denotes the scalar curvature of the metric g and $\overline{R}_g$ stands for the average $\left(\int_M dv_g\right)^{-1}\int_M R_g dv_g$. Here, dv_g is the volume form of (M, g). Although the Yamabe problem was solved using a different approach in [7,29,37], the Yamabe flow is a natural geometric deformation to metrics of constant scalar curvature. The convergence of the Yamabe flow on closed manifolds was studied in [17,33,40]. This question was completed solved in [10,11], where the author makes use of the positive mass theorem.

In this work, we study the convergence of a similar flow on compact n-dimensional manifolds with boundary, when $n \geq 3$. For those manifolds, J. Escobar raised the question of existence of conformal scalar-flat metrics on M which have the boundary as a constant mean curvature hypersurface. This problem was studied in [1,20,22,25,26,3,15]. (The question of existence of conformal metrics with constant scalar curvature and minimal boundary was studied in [12,19]; see also [6,24].)

Let (M^n, g_0) be a compact Riemannian manifold with boundary ∂M and dimension $n \geq 3$. We consider the following conformal invariant defined in [20]:

$$
\begin{aligned}
Q(M,\partial M) &= \inf_{g\in[g_0]} \frac{\int_M R_g dv_g + 2\int_{\partial M} H_g d\sigma_g}{\left(\int_{\partial M} d\sigma_g\right)^{\frac{n-2}{n-1}}} \\
&= \inf_{\{u\in C^1(\bar{M}), u\not\equiv 0 \text{ on } \partial M\}} \frac{\int_M \left(\frac{4(n-1)}{n-2}|du|_{g_0}^2 + R_{g_0}u^2\right) dv_{g_0} + \int_{\partial M} 2H_{g_0}u^2 d\sigma_{g_0}}{\left(\int_{\partial M} |u|^{\frac{2(n-1)}{n-2}} d\sigma_{g_0}\right)^{\frac{n-2}{n-1}}},
\end{aligned}
$$

where H_g and $d\sigma_g$ denote respectively the trace of the 2nd fundamental form and the volume form of ∂M, with respect to the metric g, and $[g_0]$ stands for the conformal class of the metric g_0. Although we always have $Q(M, \partial M) \leq Q(B^n, \partial B)$, where B^n is the closed unit ball in $\mathbb{R}^n$, we may have $Q(M, \partial M) = -\infty$ (see [21]).

Conformal scalar-flat metrics in compact manifolds with boundary can be easily obtained under the hypothesis that $Q(M, \partial M) > -\infty$ (which is the case when the scalar curvature is non-negative). To that end, we can use, as the conformal factor, the first eigenfunction of a linear eigenvalue problem (see [20, Proposition 1.4]).

We are interested in a formulation of a Yamabe-type flow for compact scalar-flat manifolds with boundary proposed by S. Brendle in [9]. This flow evolves a conformal family of metrics $g(t)$, $t \geq 0$, according to the equations

$$
\begin{cases}
R_{g(t)} = 0\,, & \text{in } M\,,\\
\frac{\partial}{\partial t} g(t) = -2(H_{g(t)} - \overline{H}_{g(t)})g(t)\,, & \text{on } \partial M\,,
\end{cases}
\tag{1.1}
$$

where, $\overline{H}_g$ stands for the average $\left(\int_{\partial M} d\sigma_g\right)^{-1}\int_{\partial M} H_g d\sigma_g$. (We refer the reader to Section 2 for the formulation in terms of the conformal factor.)

Brendle proved short-time existence of a unique solution to (1.1) for a given initial metric and the following long-time result: