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Existence of radial solutions to biharmonic k-Hessian equations ☆

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Abstract

This work presents the construction of the existence theory of radial solutions to the elliptic equation

$$\Delta^2 u = (-1)^k S_k[u] + \lambda f(x), \qquad x \in B_1(0) \subset \mathbb{R}^N,$$

provided either with Dirichlet boundary conditions

$$u = \partial_n u = 0, \qquad x \in \partial B_1(0),$$

or Navier boundary conditions

$$u = \Delta u = 0, \qquad x \in \partial B_1(0),$$

where the k-Hessian $S_k[u]$ is the k-th elementary symmetric polynomial of eigenvalues of the Hessian matrix and the datum $f \in L^1(B_1(0))$ while $\lambda \in \mathbb{R}$. We prove the existence of a Carathéodory solution to these boundary value problems that is unique in a certain neighborhood of the origin provided $|\lambda|$ is small enough. Moreover, we prove that the solvability set of λ is finite, giving an explicity bound of the extreme value.

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1. Introduction

This work is devoted to the study of the existence of radial solutions to elliptic equations of the form

$$\Delta^2 u = (-1)^k S_k[u] + \lambda f(x), \qquad x \in B_1(0) \subset \mathbb{R}^N, \tag{1}$$

where $N, k \in \mathbb{N}$, $\lambda \in \mathbb{R}$ and $f : B_1(0) \subset \mathbb{R}^N \longrightarrow \mathbb{R}$ is an absolutely integrable function. The first term on the right hand side of (1) is the k-Hessian $S_k[u] = \sigma_k(\Lambda)$, where

$$\sigma_k(\Lambda) = \sum_{i_1 < \cdots < i_k} \Lambda_{i_1} \cdots \Lambda_{i_k},$$

is the k-th elementary symmetric polynomial and $\Lambda = (\Lambda_1, \cdots, \Lambda_n)$ is the set of eigenvalues of the Hessian matrix $(D^2 u)$. In other words, $S_k[u]$ is the sum of the k-th principal minors of the Hessian matrix. We will always focus on the range $2 \le k \le N$, since Eq. (1) is linear for $k = 1$, and we are interested in nonlinear boundary value problems.

The motivation to study Eq. (1) comes from different sources. In the first place we can cite the impressive development of analytical results concerning the fully nonlinear boundary value problems

$$S_k[u] = f,$$

as well as related problems, that has appeared in the last decades [6,8,25,26,29–41]. Particular cases of the k-Hessian equation include the Poisson equation

$$-\Delta u = f,$$

for $k = 1$, and the Monge–Ampère equation [4,5]

$$\det(D^2 u) = f,$$

for $k = N$.

This work is also motivated by the theory of biharmonic boundary value problems. Although they have been studied much less frequently than their harmonic equivalents, they are present in many different applications and possess an inherent theoretical interest. The current knowledge of fourth order elliptic equations has considerably grown in recent times [24], but still it is not comparable to the stage of development of the theory concerning harmonic boundary value problems. Biharmonic boundary value problems studied so far include different nonlinearities, see for