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A non-homogeneous elliptic problem dealing with the level set formulation of the inverse mean curvature flow

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Abstract

In the present paper we study the Dirichlet problem for the equation

$$-\operatorname{div}\left(\frac{Du}{|Du|}\right) + |Du| = f$$

in an unbounded domain $\Omega \subset \mathbb{R}^N$, where the datum f is bounded and nonnegative. We point out that the only hypothesis assumed on $\partial\Omega$ is that of being Lipschitz-continuous. This problem is the non-homogeneous extension of the level set formulation of the inverse mean curvature flow in a Euclidean space. We introduce a suitable concept of weak solution, for which we prove existence, uniqueness and a comparison principle.

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1. Introduction

The aim of this paper is to study the problem

$$\begin{cases} -\operatorname{div}\left(\dfrac{Du}{|Du|}\right) + |Du| = f\,, & \text{in } \Omega\,; \\ u = 0\,, & \text{on } \partial E_0\,; \\ \lim\limits_{|x|\to\infty} u(x) = +\infty\,; \end{cases} \tag{1.1}$$

where $\Omega = \mathbb{R}^N \backslash \overline{E_0}$, being $N \geq 2$ and E_0 an open bounded set having Lipschitz-continuous boundary, and $0 \leq f \in L^\infty(\Omega)$. We introduce a natural concept of weak solution and prove existence, uniqueness and a comparison principle. In bounded domains, the Dirichlet problem for that equation has been considered in [15]. We will use some of the techniques introduced in this paper, but we remark that the proofs of the present paper are more involved due to the unbounded character of the domain. On the other hand, previous results in unbounded domains have dealt with the homogeneous equation (the level set formulation of the inverse mean curvature flow) assuming additional conditions of smoothness on the boundary (see [13,14,17,18]). We improve those papers in the sense that we do not assume the boundary being C^1, only Lipschitz-continuous. Nevertheless, when this article was being completed we learned of a preprint by Moser [19] in which this flow is studied from a very general perspective: it is shown that there exists a solution under the only assumption on the initial condition E_0 of being open and bounded. Thus, the present paper is actually an extension of the inverse mean curvature flow in a Euclidean space to the inhomogeneous case and using a different concept of solution. In our inhomogeneous case, the datum f plays the role of damping the inverse mean curvature flow.

The inverse mean curvature flow is a one-parameter family of hypersurfaces $\{\Gamma_t\}_{t\geq 0} \subset \mathbb{R}^N$ ($N \geq 2$) whose normal velocity $V_n(t)$ at each time t equals to the inverse of its mean curvature $H(t)$. If we let $\Gamma_t := F(\Gamma_0, t)$, then the parametric description of the inverse mean curvature flow is to find $F : \Gamma_0 \times [0, T] \to \mathbb{R}^N$ such that

$$\frac{\partial F}{\partial t} = \frac{\nu}{H}, \qquad t \geq 0, \tag{1.2}$$

where ν denotes the unit outward normal to Γ_t.

The inverse mean curvature flow and related geometric evolution problems have been studied by several authors. Among the pioneers works should be quoted [24,12,11,23]. Huisken and Ilmanen in [13] propose a level set formulation of the inverse mean curvature flow (1.2), and define a notion of weak solution using an energy minimizing principle in such a way that the generalized inverse mean curvature flow exists for all time. Using this result they then give a proof of the Penrose Inequality, which says that the total mass of a space–time containing black holes with event horizons of the total area A should be at least $\sqrt{A(16\pi)^{-1}}$, for the particular case of a single black hole.

The level set formulation proposed in [13] can be stated as follows. Assume that the flow is given by the level sets of a Lipschitz function $u : \mathbb{R}^N \to \mathbb{R}$ via

$$\Gamma_t = \partial E_t, \quad E_t := \{x \in \mathbb{R}^N \,:\, u(x) < t\}.$$