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Mild solutions to the time fractional Navier–Stokes equations in $\mathbb{R}^N$ ☆

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Abstract

This paper addresses the existence and uniqueness of mild solutions to the Navier–Stokes equations with time fractional differential operator of order $\alpha \in (0, 1)$. Several interesting properties about the solution are also highlighted, like regularity and decay rate in Lebesgue spaces, which will depend on the fractional exponent α. Moreover, it is shown that the L^p-exponent range, which the solution belongs to, is different from the range for the solution of the classical problem with $\alpha = 1$.

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Keywords: Fractional differential equations; Navier–Stokes equations; Mild solution; Mittag–Leffler functions; Decay properties

1. Introduction

Navier–Stokes equations have been extensively studied over the last century due to their importance in fluid mechanics and turbulence. The Cauchy problem for the incompressible

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Navier–Stokes equations in $\mathbb{R}^N$, $N \geq 2$, is given by

$$\begin{aligned} u_t - \nu\Delta u + (u\cdot\nabla)u + \nabla p &= f && \text{in } \mathbb{R}^N,\ t>0, \\ \nabla\cdot u &= 0 && \text{in } \mathbb{R}^N,\ t>0, \\ u(x,0) &= u_0 && \text{in } \mathbb{R}^N, \end{aligned} \tag{NS}$$

where $u = (u_1(x,t), u_2(x,t), \dots, u_N(x,t))$ represents the velocity field, $\nu > 0$ is the viscosity coefficient, $p = p(x,t)$ is the associated pressure, $u_0 = u_0(x)$ is the initial velocity and $f = (f_1(x,t), f_2(x,t), \dots, f_N(x,t))$ is an external force.

Between the many notable results regarding these equations, it is worth to stress the importance of the pioneer work of Leray [23], who proposed the matter of determining whether the solution of (NS) decays to zero in L^2 when time goes to infinity. His work motivated many researchers to address this issue and, nowadays, there is a wide and solid literature on this subject; we may cite, among others, [1,2,6,18,33,38,37]. Therewith, it is reasonable to conclude that the decay properties of this problem are already well understood.

On the other hand, having its beginning since the birth of differential calculus and being subject of discussion for many notable mathematicians like Leibniz, l'Hôpital, Euler, Fourier, Abel, Liouville, and Riemann, fractional calculus has been extensively studied achieving a great success in the last four decades. For a long time, fractional calculus has been regarded as a pure mathematical tool without real applications. But, in recent decades, it has been found that fractional calculus can be useful in the most diverse areas of science, mainly due to the nonlocal character of the fractional differentiation. Among the countless applications of the fractional calculus, it is worth mentioning some solid works on stochastic processes driven by fractional Brownian motion [39] and on physical phenomena like electromagnetism [13] and viscoelasticity [5,10,17]. For more examples, we refer to the survey [26] and references therein.

In this context, it is natural that there exists an interest in relating these two previous subjects. Indeed, in [34] Shinbrot considered weak solutions of the Navier–Stokes equations and established several lemmas on the regularity of its fractional derivative. However, it was only Zhang in [40] that managed to prove Shinbrot's conjecture, which stated that any weak solution of the Navier–Stokes equations possesses any fractional derivatives of order less than or equal to $1/2$.

Consequently, it is not surprising that very recently a new discussion about this matter has started proposing generalized Navier–Stokes equations with time fractional differential operator:

$$\begin{aligned} cD_t^\alpha u - \nu\Delta u + (u\cdot\nabla)u + \nabla p &= f && \text{in } \mathbb{R}^N,\ t>0, \\ \nabla\cdot u &= 0 && \text{in } \mathbb{R}^N,\ t>0, \\ u(x,0) &= u_0 && \text{in } \mathbb{R}^N, \end{aligned} \tag{FNS}$$

where $\alpha \in (0,1)$ is a fixed number and cD_t^α is the Caputo fractional derivative (see Definition 1); for more details we refer to [12,16,19,29,30].

Notice that the exchange of the time differential operator by the time fractional differential operator plays the role of a new parameter in the problem which provides new insights about the whole scenario.

In this paper we are interested in performing a rigorous study of mild solutions to the Navier–Stokes equations with time fractional derivative (FNS). To this end, a suitable integral formulation to this problem and a proper definition of mild solution is introduced (see Definition 4). This kind of solution involves a family of operators called Mittag–Leffler, which arises naturally