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Conditional entropy and fiber entropy for amenable group actions ☆

Kesong Yan [a,b,*]

[a] Department of Mathematics, University of Science and Technology of China, Hefei, Anhui 230026, PR China

[b] School of Information and Statistics, Guangxi University of Finance and Economics, Nanning, Guangxi 530003, PR China

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Abstract

In this paper, we introduce the notions of topological conditional entropy and fiber entropy for a given factor map between two amenable group actions, and prove three variational principles for conditional entropy and fiber entropy. Moreover, as an application of our variational principle we prove that the countable-to-one extension and the distal extension have zero conditional topological entropy. In particular, the topological entropy of a distal system is zero.

MSC: primary 37A35; secondary 37B05, 37B40

Keywords: Conditional entropy; Fiber entropy; Amenable group actions; Variational principle; Distal system

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* Correspondence to: School of Information and Statistics, Guangxi University of Finance and Economics, Nanning, Guangxi, 530003, PR China.

E-mail address: ksyan@mail.ustc.edu.cn.

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1. Introduction

In this paper, we introduce several notions of topological conditional entropy and fiber entropy for a given factor map between two amenable group actions and study their relationships with the measure-theoretic conditional entropy. We also study the topological conditional entropy of countable-to-one extensions and distal extensions. Firstly, we discuss the motivations for this topic.

1.1. Motivations

The classical measure-theoretic entropy for an invariant measure was first introduced by Kolmogorov [40] in 1958 and later by Sinai [59] in the general case. The classical topological entropy was first introduced by Adler, Konheim and McAndrew [1] in 1965, and later Dinaburg [13] and Bowen [9] gave two equivalent definitions by using separated and spanning sets.

For each dynamical system (X, T) of $\mathbb{Z}$-actions (or called it TDS), Goodwyn (1969) [26] proved the inequality $h_\mu(T) \leqslant h_{\mathrm{top}}(T)$ between measure-theoretic and topological entropy. Later, Goodman [25] showed that $\sup_\mu h_\mu(T) \geqslant h_{\mathrm{top}}(T)$, where the supremum is taken over all invariant measures, completing the classical variational principle. A short and elegant proof of the variational principle for $\mathbb{Z}_+^d$-actions was given by Misiurewicz [47]. Since then a subject involving definition of new measure-theoretic and topological notions of entropy and the relationship between them has gained a lot of attention in the study of dynamical systems.

In 1971, Bowen [9] considered a factor map π between two TDSs (X, T) and (Y, S), and formulated a version of topological conditional variational principle, that is,

$$h_{\mathrm{top}}(T) \leqslant h_{\mathrm{top}}(S) + \sup_{y\in Y} h_{\mathrm{top}}(T, \pi^{-1}(y)),$$

where $h_{\mathrm{top}}(T, K)$ is the topological entropy of a compact subset K of X.

In 1976, Misiurewicz [48] introduced the notion of topological conditional entropy of a topological dynamical system, and showed it is bigger than “the defect of upper semi-continuity” of the measure-theoretic entropy function defined on the space of invariant measures. Generally, it is strictly bigger (see [48, Example 6.4]). Later, Ledrappier [41] showed that Misiurewicz’s conditional entropy can be defined by means of the entropy function on the cartesian square of the system. The formula looks like a variational principle for the topological conditional entropy.

In 1977, Ledrappier and Walters [42] worked in the setting of Bowen [9] and proved a relative variational principle for a given factor map $\pi : (X, T) \to (Y, S)$:

$$\sup_{\mu:\pi\mu=\nu} h_\mu(T) = h_\nu(S) + \int_Y h_{\mathrm{top}}(T, \pi^{-1}(y))\, \mathrm{d}\nu(y).$$

Moreover, Downarowicz and Serafin [17] introduced the notions of relative topological entropy $h_{\mathrm{top}}(T, X|Y)$ and relative measure-theoretical entropy $h_\mu(T, X|Y)$ for an invariant measure μ, and proved a more general relative variational principle:

$$h_{\mathrm{top}}(T, X|Y) = \sup_{y\in Y} h_{\mathrm{top}}(T, \pi^{-1}(y)) = \sup_\mu h_\mu(T, X|Y).$$