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A reaction–diffusion–advection model of harmful algae growth with toxin degradation

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Abstract

This paper is devoted to the study of a reaction–diffusion–advection system modeling the dynamics of a single nutrient, harmful algae and algal toxin in a flowing water habitat with a hydraulic storage zone. We introduce the basic reproduction ratio $\mathcal{R}_0$ for algae and show that $\mathcal{R}_0$ serves as a threshold value for persistence and extinction of the algae. More precisely, we prove that the washout steady state is globally attractive if $\mathcal{R}_0 < 1$, while there exists a positive steady state and the algae is uniformly persistent if $\mathcal{R}_0 > 1$. With an additional assumption, we obtain the uniqueness and global attractivity of the positive steady state in the case where $\mathcal{R}_0 > 1$.

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Keywords: Spatial model of harmful algae; Steady states; Basic reproduction ratio; Threshold dynamics; Global attractor

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1. Introduction

Blooms of the harmful algae have increased the intensity worldwide in coastal as well as inland waters. The blooms have direct impacts for human health, and food webs in aquatic ecosystems [3]. For example, *Prymnesium parvum* (golden algae) is responsible for such harmful algal blooms worldwide that have caused large fish kills and millions of dollars in economic losses. There is a paradox in the persistence of harmful algae [15]. Intuitively, strong flow washes out suspended algae, and continual strong flow can overcome the reproductive capacity of planktonic algae. On the other hand, the characteristics of the shorelines and the bed of the channel can reduce the speed of flow, producing slow-flowing regions constituting a hydraulic storage zone that affects the persistence of harmful algae and their toxins [3,4]. Thus, we may expect that the reproductive capacity of algae may suffice to permit population growth, even to bloom proportions. This prediction was confirmed by Grover et al. [4, Section 3.3]. Recently, a potential technique was suggested to manage and mitigate harmful algal blooms through flow manipulations in some riverine systems [8,9,17]. This possibility motivates the theoretical modeling of harmful algal dynamics in flowing habitats [3]. In order to investigate the differences between a fringing cove and a main lake arising in a single cove, Grover et al. [3] proposed two-compartment models in which one compartment is a small cove connected to a larger lake. Hsu et al. [5] further analyzed such a two-compartment model with seasonal temperature variations.

To understand longitudinal patterns arising along the axis of flow, the authors in [3] proposed two reaction–diffusion–advection systems modeling the dynamics of one nutrient, one single population of algae, and algal toxin with spatial variations in an idealized riverine reservoir where a main channel was coupled to a hydraulic storage zone. Next, we shall adopt notations and physical settings used in [3] to describe the model systems. Suppose that L represents the length of the channel; A and A_S represent cross-section area of a flowing zone, and a static storage zone, respectively. We assume that advective and diffusive transport occur only in the main flowing zone, not the storage zone; α (time^{-1}) represents the exchange rate of nutrient, algae, and toxin between the flowing and storage zones. Flow enters at the upstream end of the channel ($x = 0$), and an equal flow exits at the downstream end ($x = L$). Flow is parameterized as a constant dilution rate D (time^{-1}), and assuming constant water volume in the channel implies that advection occurs at a speed ν ($\nu = DL$). The flow of water in the channel in the direction of increasing x brings fresh nutrient for algal growth at a concentration $R^{(0)}$ into the reactor at $x = 0$, and a balancing flow exits at the dam ($x = L$), removing algae, nutrients, and algal toxin. Nutrient, algae, and algal toxin are assumed to diffuse throughout the main channel with the same diffusivity δ. Both advective and diffusive transport occur at the upstream boundary ($x = 0$). The downstream boundary is assumed to be a dam, over which there is advective flow but through which no diffusion can take place.

The nonlinear function $f(R)$ describes the nutrient uptake and algal growth at the limiting nutrient concentration (R). We assume that $f(R)$ satisfies

$$f(0) = 0, \ f'(R) > 0, \ f \in C^2.$$

A typical example is the Monod function

$$f(R) = \frac{\mu_{\max} R}{K + R},$$