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Remark on the scattering operator for the cubic nonlinear Dirac equation in three space dimensions

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Abstract

This paper is concerned with the scattering operator S for the three-dimensional Dirac equation with a cubic nonlinearity. It follows from known results that S is well-defined on a neighborhood of 0 in the Sobolev space $H^\kappa(\mathbb{R}^3;\mathbb{C}^4)$ for any $\kappa > 1$. In the present paper, we prove that for any $M \in \mathbb{N}$ and $s \geq \max\{\kappa, M\}$, there exists some neighborhood U of 0 in the weighted Sobolev space $H^{s,M}(\mathbb{R}^3;\mathbb{C}^4)$ such that $S(U) \subset H^{s,M}(\mathbb{R}^3;\mathbb{C}^4)$.

MSC: 35P25; 35Q55; 35G25

Keywords: Nonlinear Dirac equation; Scattering operator; Weighted Sobolev space

1. Introduction

In this paper, we study scattering problems for the nonlinear Dirac equation in three space dimensions

$$(\partial_t + \alpha \cdot \nabla + i\beta)\,\Phi = \mathcal{N}(\Phi). \tag{NLD}$$

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Here, $\Phi = \Phi(t, x)$ is a $\mathbb{C}^4$-valued unknown function defined on $\mathbb{R} \times \mathbb{R}^3$, $\partial_t = (\partial/\partial t)I$, I is the 4×4 identity matrix, $i = \sqrt{-1}$, $\alpha \cdot \nabla = \alpha_1\partial_1 + \alpha_2\partial_2 + \alpha_3\partial_3$, $\partial_j = (\partial/\partial x_j)I$ $(j = 1, 2, 3)$, α_1, α_2, α_3 and β are 4×4 Hermitian matrices satisfying that

$$\alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{jk}I, \quad \alpha_j\beta + \beta\alpha_j = 0, \quad \beta^2 = I \ (1 \le j, k \le 3),$$

and $\mathcal{N}$ is a mapping from $\mathbb{C}^4$ into itself. Throughout this paper we assume that $\mathcal{N}$ is a homogeneous cubic polynomial in z and its conjugate $z^\dagger$ $(z \in \mathbb{C}^4)$. Namely, we assume the following condition:

(N) Some $\mathbb{C}^4$-valued constants Λ_a $(a \in (\mathbb{N} \cup \{0\})^8$ with $|a| = 3)$ satisfy that

$$\mathcal{N}(z) = \sum_{|a|=3} (z, z^\dagger)^a \Lambda_a, \quad z \in \mathbb{C}^4.$$

The functions

$$\mathcal{N}_0(z) = \lambda(z|z)z$$

and

$$\mathcal{N}_1(z) = \lambda(\beta z|z)\beta z$$

are typical examples satisfying (N), where $\lambda \in \mathbb{C}$ and $(\cdot|\cdot)$ is the $\mathbb{C}^4$-inner product.

Eq. (NLD) with $\mathcal{N} = \mathcal{N}_1$ is one of the most important models in relativistic quantum fields and the Cauchy problem associated with (NLD) was studied (see, e.g., [2,4,9,10] and references therein). In particular, Machihara, Nakanishi and Ozawa [10] proved that if $\mathcal{N} = \mathcal{N}_1$, $\kappa > 1$ and $\Phi_0 \in H^\kappa$ is sufficiently small, then there exist a unique time-global solution $\Phi \in Y$ to (NLD) with the initial condition $\Phi(0) = \Phi_0$ and a unique data $\Phi_+ \in H^\kappa$ such that

$$\lim_{t \to +\infty} \left\| \Phi(t) - D(t)\Phi_+; H^\kappa \right\| = 0. \tag{1.1}$$

Here, H^κ denotes the Sobolev space $\omega^{-\kappa}L^2(\mathbb{R}^3; \mathbb{C}^4)$, $\omega = \sqrt{1 - \Delta}$, Y is a given subspace of $C(\mathbb{R}; H^\kappa)$ and $D(t)\Phi_+$ denotes the solution to the free Dirac equation $(\partial_t + \alpha \cdot \nabla + i\beta)\Phi = 0$ with the initial condition $\Phi(0) = \Phi_+$ (for the precise definition of $D(t)$, see (1.2) below). Therefore, the inverse wave operator $V_+ : \Phi_0 \mapsto \Phi_+$ is well-defined as a mapping from some neighborhood of 0 in H^κ. Similarly, we can immediately see that for any $\mathcal{N}$ satisfying (N), the inverse wave operator, the wave operator $W_- : \Phi_- \mapsto \Phi(0)$ and the scattering operator $S = V_+ \circ W_-$ for (NLD) are well-defined (for the precise statement of the existence of W_-, V_+ and S, see Proposition 1.2 below).

Remark 1.1. In the case $\mathcal{N} = \mathcal{N}_1$, recently Bejenaru and Herr [1] proved that (NLD) is globally well-posed for small initial data in H^1 and small solutions scatter to free solutions as $t \to \infty$. In the proof, they used a null structure in a system of equations which is equivalent to (NLD) with $\mathcal{N} = \mathcal{N}_1$. It does not seem that for any $\mathcal{N}$ satisfying (N) the method in [1] is applicable to (NLD).