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The turnpike property in finite-dimensional nonlinear optimal control

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Abstract

Turnpike properties have been established long time ago in finite-dimensional optimal control problems arising in econometry. They refer to the fact that, under quite general assumptions, the optimal solutions of a given optimal control problem settled in large time consist approximately of three pieces, the first and the last of which being transient short-time arcs, and the middle piece being a long-time arc staying exponentially close to the optimal steady-state solution of an associated static optimal control problem. We provide in this paper a general version of a turnpike theorem, valuable for nonlinear dynamics without any specific assumption, and for very general terminal conditions. Not only the optimal trajectory is shown to remain exponentially close to a steady-state, but also the corresponding adjoint vector of the Pontryagin maximum principle. The exponential closedness is quantified with the use of appropriate normal forms of Riccati equations. We show then how the property on the adjoint vector can be adequately used in order to initialize successfully a numerical direct method, or a shooting method. In particular, we provide an appropriate variant of the usual shooting method in which we initialize the adjoint vector, not at the initial time, but at the middle of the trajectory.

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1. Introduction and main result

Dynamical optimal control problem. Consider the nonlinear control system

$$\dot{x}(t) = f\big(x(t), u(t)\big), \tag{1}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is of class C^2. Let $R = (R^1, \dots, R^k) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^k$ be a mapping of class C^2, and let $f^0 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be a function of class C^2. For a given $T > 0$ we consider the optimal control problem $\mathbf{(OCP)_T}$ of determining a control $u_T(\cdot) \in L^\infty(0, T; \mathbb{R}^m)$ minimizing the cost functional

$$C_T(u) = \int_0^T f^0\big(x(t), u(t)\big)\, dt \tag{2}$$

over all controls $u(\cdot) \in L^\infty(0, T; \mathbb{R}^m)$, where $x(\cdot)$ is the solution of (1) corresponding to the control $u(\cdot)$ and such that

$$R\big(x(0), x(T)\big) = 0. \tag{3}$$

We assume throughout that $\mathbf{(OCP)_T}$ has at least one optimal solution $(x_T(\cdot), u_T(\cdot))$. Conditions ensuring the existence of an optimal solution are well known (see, e.g., [17,51]). For example, if the set of velocities $\{f(x, u) \mid u \in \mathbb{R}^m\}$ is a convex subset of $\mathbb{R}^n$ for every $x \in \mathbb{R}^n$, with mild growth at infinity, and if the epigraph of f^0 is convex, then there exists at least one optimal solution. Note that this is the case whenever the system (1) is control-affine, that is, $f(x, u) = f_0(x) + \sum_{i=1}^m u_i f_i(x)$, where the f_i's, $i = 0, \dots, m$, are C^1 vector fields in $\mathbb{R}^n$ growing mildly at infinity, and f^0 is a positive definite quadratic form in (x, u). The classical linear quadratic problem fits in this class (and in that case the optimal solution is moreover unique).

According to the Pontryagin maximum principle (see [2,41,51]), there must exist an absolutely continuous mapping $\lambda_T(\cdot) : [0, T] \to \mathbb{R}^n$, called *adjoint vector*, and a real number $\lambda_T^0 \leqslant 0$, with $(\lambda_T(\cdot), \lambda_T^0) \neq (0, 0)$, such that, for almost every $t \in [0, T]$,

$$\begin{aligned}
\dot{x}_T(t) &= \frac{\partial H}{\partial \lambda}\big(x_T(t), \lambda_T(t), \lambda_T^0, u_T(t)\big), \\
\dot{\lambda}_T(t) &= -\frac{\partial H}{\partial x}\big(x_T(t), \lambda_T(t), \lambda_T^0, u_T(t)\big), \\
\frac{\partial H}{\partial u}&\big(x_T(t), \lambda_T(t), \lambda_T^0, u_T(t)\big) = 0,
\end{aligned} \tag{4}$$

where the Hamiltonian H of the optimal control problem is defined by

$$H\big(x, \lambda, \lambda^0, u\big) = \big\langle \lambda, f(x, u)\big\rangle + \lambda^0 f^0(x, u), \tag{5}$$