Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 257 (2014) 3079–3101

Journal of Differential Equations

www.elsevier.com/locate/jde

# Varieties of local integrability of analytic differential systems and their applications

Valery G. Romanovski [a], Yonghui Xia [b,*], Xiang Zhang [c]

[a] Center for Applied Mathematics and Theoretical Physics, and Faculty of Natural Science and Mathematics, University of Maribor, SI-2000 Maribor, Slovenia

[b] Department of Mathematics, Zhejiang Normal University, Jinhua 321004, Zhejiang, People's Republic of China

[c] Department of Mathematics, and MOE-LSC, Shanghai Jiao Tong University, Shanghai 200240, People's Republic of China

Received 21 August 2013

Available online 18 June 2014

## Abstract

In this paper we provide a characterization of local integrability for analytic or formal differential systems in $\mathbb{R}^n$ or $\mathbb{C}^n$ via the integrability varieties. Our result generalizes the classical one of Poincaré and Lyapunov on local integrability of planar analytic differential systems to any finitely dimensional analytic differential systems. As an application of our theory we study the integrability of a family of four-dimensional quadratic Hamiltonian systems.


*MSC:* 34A25; 34C14; 34C20; 37C15

*Keywords:* Analytic differential systems; Local integrability; Variety; Hamiltonian system; Darboux integrability

## 1. Introduction and the main results

The theory on local integrability of analytic or formal differential systems is classical and it can be traced back to H. Poincaré. This theory is very useful in the study of local dynamics of

* Corresponding author.

*E-mail addresses:* valery.romanovsky@uni-mb.si (V.G. Romanovski), xiadoc@163.com (Y. Xia), xzhang@sjtu.edu.cn (X. Zhang).

http://dx.doi.org/10.1016/j.jde.2014.06.007

dynamical systems, see e.g. [1,3,16]. By the fundamental theorem which goes back to Poincaré and Lyapunov a nondegenerate monotone singularity of a real planar analytic differential system

$$\dot{u} = -v + p(u, v), \qquad \dot{v} = u + q(u, v), \tag{1}$$

where $p$ and $q$ are convergent series without free and linear terms, is a center if and only if the system is locally analytically equivalent to a system of the form

$$\dot{x} = ix\big(1 + g(xy)\big), \qquad \dot{y} = -iy\big(1 + g(xy)\big),$$

where, $i = \sqrt{-1}$, $x = u + iv$ and $y$ is complex conjugate to $x$.

This classical theory of Poincaré and Lyapunov was recently generalized to analytic differential systems of any finite dimension $n \in \mathbb{N}$, see e.g. [12,19,20,22], where the local analytic integrability was characterized by the number of linearly independent resonances of eigenvalues of the linearized matrix and the existence of analytic normalization for reducing the analytic differential systems to its Poincaré–Dulac normal form.

Poincaré also proved that for a real planar analytic differential system (1) there exists an analytic function or a formal series $\Phi(u, v)$ such that

$$\big(-v + p(u, v)\big)\frac{\partial \Phi}{\partial u} + \big(u + q(u, v)\big)\frac{\partial \Phi}{\partial v} = \sum_{l=m}^{\infty} \alpha_l \big(u^2 + v^2\big)^l,$$

where $2 \le m \in \mathbb{N}$, and $\alpha_l$ are polynomials in the coefficients of $p(u, v)$ and $q(u, v)$. This paper will pursue the study along this direction on higher dimensional analytic or formal differential systems.

Consider the local analytic or formal differential systems

$$\dot{x} = Ax + \mathbf{f}(x), \quad \text{in } \big(\mathbb{F}^n, 0\big),\ \mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}, \tag{2}$$

where $A \in \mathrm{Mat}(\mathbb{F}, n)$, $x = (x_1, \dots, x_n)^\tau$, $\mathbf{f}(x) = (f_1(x), \dots, f_n(x))^\tau$, and $f_i$ are series (convergent or not) starting with at least quadratic terms. Recall that $\mathrm{Mat}(\mathbb{F}, n)$ denotes the set of all matrices of order $n$ with entries in $\mathbb{F}$, and $\tau$ denotes the transpose of a matrix.

Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be the $n$-tuple of eigenvalues of $A$. Set $\mathbb{Z}_+ = \mathbb{N} \cup 0$. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$ denote $\langle \lambda, \alpha \rangle = \sum_{i=1}^n \alpha_i \lambda_i$ and $|\alpha| = \alpha_1 + \ldots + \alpha_n$. Let

$$\mathfrak{R} = \big\{\alpha \in \mathbb{Z}_+^n \bigm| \langle \lambda, \alpha \rangle = 0,\ |\alpha| > 0\big\},$$

and denote by $r_\lambda$ the rank of vectors in the set $\mathfrak{R}$.

As it is well-known, see e.g. [1,16,18], there is a substitution tangent to identity

$$x = \Phi(y) := y + \varphi(y), \tag{3}$$

with $\varphi(y)$ being a series without constant and linear terms, which transforms system (2) to its Poincaré–Dulac normal form, i.e. a system of the form

$$\dot{y} = Ay + \mathbf{g}(y), \tag{4}$$