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Semilinear biharmonic problems with a singular term ☆

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Abstract

The aim of this work is to study the optimal exponent p to have solvability of problem

$$\begin{cases} \Delta^2 u = \lambda \dfrac{u}{|x|^4} + u^p + cf & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = -\Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $p > 1$, $\lambda > 0$, $c > 0$, and $\Omega \subset \mathbb{R}^N$, $N > 4$, is a smooth and bounded domain such that $0 \in \Omega$.

We consider $f \geqslant 0$ under some hypothesis that we will precise later.

MSC: 35J91; 35B25

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1. Introduction

The main concern of this work is to determine a critical threshold exponent p to have solvability of the following problem,

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$$\begin{cases} \Delta^2 u = \lambda \dfrac{u}{|x|^4} + u^p + cf & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = -\Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $p > 1$, $\lambda > 0$, $c > 0$, and $\Omega \subset \mathbb{R}^N$, $N > 4$, is a smooth and bounded domain such that $0 \in \Omega$.

There exists a large literature dealing with the case $\lambda = 0$. The differential equation $\Delta^2 u = f$ is called the Kirchhoff–Love model for the vertical deflection of a thin elastic plate. For a more elaborate history of the biharmonic problem and the relation with elasticity from an engineering point of view one may consult a survey of Meleshko [19]. In particular, the Kirchhoff–Love model with Navier conditions, $u = -\Delta u = 0$ on $\partial\Omega$, corresponds to the hinged plate model and allows rewriting these fourth-order problems as a second-order system.

Among nonlinear problems for fourth-order elliptic equations with $\lambda = 0$, there exist several results concerning semilinear equations with power type nonlinear sources (see for example [17]). A crucial role is played by the critical (Sobolev) exponent, $\frac{N+4}{N-4}$, which appears whenever $N > 4$ (see [5]).

The case $\lambda > 0$ is quite different. The singular term $\frac{u}{|x|^4}$ is related to the Hardy inequality:

Let $u \in C^2(\Omega)$ and $N > 4$, then it holds that

$$\Lambda_N \int_\Omega \frac{u^2}{|x|^4} dx \leqslant \int_\Omega |\Delta u|^2 dx,$$

where $\Lambda_N = (\frac{N^2(N-4)^2}{16})$ is optimal (see Theorem 2.2 below).

First of all, it is not difficult to show that any positive supersolution of (1) is unbounded near the origin and then additional hypotheses on p are needed to ensure existence of solutions. We will say that problem (1) *blows up completely* if the solutions to the truncated problems (with the weight $\frac{\lambda}{|x|^4+\frac{1}{n}}$ instead of the Hardy type term $\frac{\lambda}{|x|^4}$) tend to infinity for every $x \in \Omega$ as $n \to \infty$.

The main objective of this work is to explain the influence of the Hardy type term on the existence or nonexistence of solutions and to determine the threshold exponent $p_+(\lambda)$ to have a complete blow-up phenomenon if $p \geqslant p_+(\lambda)$.

The corresponding elliptic semilinear case with the Laplacian operator was studied in [8,14], where the authors show the existence of a critical exponent $p_*(\lambda) > 1$ such that the problem has no local distributional solution if $p \geqslant p_*(\lambda)$. Furthermore, they prove the existence of solutions with $p < p_*(\lambda)$ under some suitable hypothesis on the datum.

In fourth-order problems, Navier boundary conditions play an important role to prove existence results. The problem can be rewritten as a second-order system with Dirichlet boundary conditions. By the classical elliptic theory, we easily prove a Maximum Principle. As a consequence, we deduce a Comparison Principle that allows us to prove the existence of solutions for $p < p_+(\lambda)$ as a limit of approximated problems.

The paper is organized as follows.

In Section 2 we briefly describe the natural functional framework for our problem and the embeddings we will use throughout the paper.

Section 3 is devoted to some definitions and preliminary results. First, we describe the radial solutions to the homogeneous problem that allow us to know the singularity of our supersolutions