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Existence and regularity of multiple solutions for infinitely degenerate nonlinear elliptic equations with singular potential ☆

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Abstract

In this paper, we study the Dirichlet problem for a class of infinitely degenerate nonlinear elliptic equations with singular potential term. By using the logarithmic Sobolev inequality and Hardy's inequality, the existence and regularity of multiple nontrivial solutions have been proved.

MSC: 35J60; 35J70

Keywords: Infinitely degenerate elliptic equations; Logarithmic Sobolev inequality; Hardy's inequality; Singular potential

1. Introduction and main results

In this paper, we study the existence and regularity of solution for the following semi-linear infinitely degenerate elliptic equation

$$\begin{cases} -\Delta_X u - \varepsilon V_n u = au \log|u| + bu + g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where vector fields $X = (\partial_{x_1}, \cdots, \partial_{x_{n-1}}, \varphi(x')\partial_{x_n})$ defined on an open domain $\tilde{\Omega} \subset \mathbb{R}^n$ for $n \geq 3$, Ω is a bounded open subset in $\tilde{\Omega}$ which contains the origin, $a > 0$, $b > 0$; $\varphi(x')$ is a non-negative C^∞-smooth function in $x' = (x_1, x_2, \cdots, x_{n-1})$ and for $\Gamma \subset \tilde{\Omega}$, $\partial^\alpha_{x'}\varphi(x')|_{(x',x_n)\in\Gamma} = 0$ for any $|\alpha| \geq 0$; $g(x,u)$ is a Carathéodory function with primitive $G(x,u) = \int_0^u g(x,v)dv$, which satisfies the following conditions:

(f_1) $g(x,u) = -g(x,-u)$,
(f_2) there exist $q \in (1,2)$, and $c > 0$, such that $|g(x,u)| \leq c(1 + |u|^{q-1})$,
(f_3) if $u \in L^2(\Omega)$ and $u \neq 0$, then $\lim_{t\to 0^+} \frac{\int_\Omega G(x,tu)dx}{t^2 \log t} = -\infty$,
(f_4) there exists $\varepsilon_0 > 0$, such that for $0 \leq u \leq \varepsilon_0$, we have $g(x,u) + au\log|u| + bu \geq 0$,
(f_5) $g(x,u)$ is C^∞ in x, and C^∞ in u except $u = 0$.

Here we suppose that the vector fields X satisfies the following logarithmic regularity estimates,

$$\left\|(\log \Lambda)^s u\right\|^2_{L^2(\Omega)} \leq C_0\left[\int_\Omega |Xu|^2 dx + \|u\|^2_{L^2(\Omega)}\right] \tag{1.2}$$

for all $u \in C_0^\infty(\tilde{\Omega})$, where $\Lambda = (e^2 + |D|^2)^{\frac{1}{2}} = \langle D\rangle$. Also the potential term $V_n(x) \geq 0$ may be unbounded in Ω and satisfies the following Hardy's inequality

$$\int_\Omega V_n u^2 dx \leq \int_\Omega |Xu|^2 dx, \quad \text{for all } u \in H^1_{X,0}(\Omega), \tag{1.3}$$

where $H^1_{X,0}(\Omega)$ is Hilbert space as defined in Section 2 below.

Let

$$M(q,\Omega) = \frac{(1-\varepsilon)\eta_1}{4}\left(\frac{1}{q}C^{2-q} + C\right)^{-1}, \tag{1.4}$$

where $C = |\Omega|e^{C_1/a}$, $C_1 = \frac{C_0 a^2}{2(1-\varepsilon)} + \frac{(1-\varepsilon)}{2} + b - \frac{a}{2}$, $|\Omega|$ is the Lebesgue measure of Ω, C_0 is the positive constant appeared in (1.2) and $\eta_1 > 0$ is the first eigenvalue of the operator $-\Delta_X$.

In this paper we need the following hypothesis,

(H-1) $\partial\Omega$ is C^∞ and non-characteristic for the system of vector fields X;
(H-2) X satisfies the finite type of Hörmander's condition with Hörmander index Q on $\tilde{\Omega}$ except a union of smooth surfaces Γ which are non-characteristic for X;
(H-3) X satisfies Logarithmic regularity estimate (1.2) with $s \geq \frac{3}{2}$.
(H-4) The non-negative singular potential function $V_n(x)$ is $C^\infty(\Omega\backslash\Gamma_1)$, here $\Gamma_1 \subseteq \Gamma$ is the set on which $V_n(x)$ is unbounded, and satisfies the Hardy's inequality (1.3).

Thus we have the following main results.

Theorem 1.1. *Under the conditions* (H-1), (H-2), (H-3) *and* (H-4), *if* $0 < \varepsilon < 1$, *and* $0 \leq \varphi(x') \leq 1$, *then we have*