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Approximate controllability for nonlinear degenerate parabolic problems with bilinear control [☆]

Giuseppe Floridia [1]

Dipartimento di Matematica, Università di Roma "Tor Vergata", I-00161 Roma, Italy

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Abstract

In this paper, we study the global approximate multiplicative controllability for nonlinear degenerate parabolic Cauchy–Neumann problems. First, we obtain embedding results for weighted Sobolev spaces, that have proved decisive in reaching well-posedness for nonlinear degenerate problems. Then, we show that the above systems can be steered in L^2 from any nonzero, nonnegative initial state into any neighborhood of any desirable nonnegative target-state by bilinear piecewise static controls. Moreover, we extend the above result relaxing the sign constraint on the initial data.

MSC: 93B05; 35K58; 35K65; 35K61; 35K45

Keywords: Approximate controllability; Bilinear control; Semilinear equations; Degenerate parabolic equations; Weighted Sobolev spaces

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E-mail addresses: floridia@mat.uniroma2.it, floridia.giuseppe@icloud.com.

[1] Post Doc *Istituto Nazionale di Alta Matematica* (INdAM) "F. Severi", Roma.

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1. Introduction

This paper is concerned with the analysis of semilinear parabolic control systems in one space dimension, governed in the bounded domain $(-1, 1)$ by means of the *bilinear control* $\alpha(t, x)$, of the form

$$\begin{cases} u_t - \big(a(x)u_x\big)_x = \alpha(t,x)u + f(t,x,u) & \text{in } Q_T := (0,T) \times (-1,1) \\ a(x)u_x(t,x)|_{x=\pm 1} = 0 & t \in (0,T) \\ u(0,x) = u_0(x) & x \in (-1,1). \end{cases} \tag{1.1}$$

The equation in the *Cauchy–Neumann* problem above is a degenerate parabolic equation, because the diffusion coefficient, positive on $(-1, 1)$, is allowed to vanish at the extreme points of $[-1, 1]$.

The main physical motivations for studying degenerate parabolic problems with the above structure come from mathematical models in climate science as we explain below.

1.1. Physical motivations: climate models and degenerate parabolic equations

Climate depends on various parameters such as temperature, humidity, wind intensity, the effect of greenhouse gases, and so on. It is also affected by a complex set of interactions in the atmosphere, oceans and continents, that involve physical, chemical, geological and biological processes.

One of the first attempts to model the effects of the interaction between large ice masses and solar radiation on climate is the one due, independently, to Budyko [8,9], and Sellers [41] (see also [20–23,30,42,4,43] and the references therein). The Budyko–Sellers model is an *energy balance model*, which studies the role played by continental and oceanic areas of ice on climate change. The effect of solar radiation on climate can be summarized in the following:

$$\textit{Heat variation} = R_a - R_e + D,$$

where R_a is the *absorbed energy*, R_e is the *emitted energy* and D is the *diffusion part*.

The general formulation of the Budyko–Sellers model on a compact surface $\mathcal{M}$ without boundary is as follows

$$u_t - \Delta_{\mathcal{M}} u = R_a(t, X, u) - R_e(t, X, u),$$

where $u(t, X)$ is the distribution of temperature, $\Delta_{\mathcal{M}}$ is the classical Laplace–Beltrami operator, $R_a(t, X, u) = Q(t, X)\beta(X, u)$. In the above, Q is the *insolation* function, that is, the incident solar radiation at the top of the atmosphere. In annual models, when the time scale is long enough, one may assume that the insolation function doesn't depend on time t, i.e. $Q = Q(X)$. But, when the time scale is smaller, as in seasonal models, one uses a more realistic description of the incoming solar flux by assuming that Q depends on t, i.e. $Q = Q(t, X)$. β is the *coalbedo* function, that is, 1-*albedo function*. Albedo is the reflecting power of a surface. It is defined as the ratio of reflected radiation from the surface to incident radiation upon it. It may also be expressed as a percentage, and is measured on a scale from zero, for no reflecting power of a perfectly black surface, to 1, for perfect reflection of a white surface.