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Spectral properties of Sturm–Liouville operators with local interactions on a discrete set ☆

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Abstract

This paper deals with the spectral properties of the Sturm–Liouville operators generated by the differential expression

$$L(y) = \frac{1}{w}\left(-\left(p(x)y'\right)' + \upsilon(x)y\right)$$

with singular coefficients $\upsilon(x)$ in the sense of distributions. In particular, we study the operators with δ-type point interactions at the centers x_k on the positive half line in terms of energy forms. Necessary and sufficient conditions and also simple sufficient conditions are given for the spectrum of the operators to be discrete. We also prove sufficient conditions on the stability of the continuous spectrum.

MSC: primary 34B24; secondary 34L05, 34L99

Keywords: Sturm–Liouville problems; Discrete spectrum; Continuous spectrum; Distribution potentials

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1. Introduction

There are several approaches for defining an operator corresponding to the Sturm–Liouville differential expression

$$L_\upsilon y = \frac{1}{w}\big(-\big(p(x)y'\big)' + \upsilon(x)y\big), \quad x \in \mathbb{R}_+, \tag{1.1}$$

with a distributional potential $\upsilon = u'$ where the derivative of the real function u is understood in the sense of distributions, i.e. $(\upsilon, \varphi) = -(u, \varphi')$ for any infinitely differentiable function $\varphi(x)$ compactly supported on the interval $(0, +\infty)$, and

$$1/p,\ u/p,\ u^2/p,\ w \in L^1_{loc}(\mathbb{R}_+), \quad p > 0,\ w > 0 \text{ a.e. on } \mathbb{R}_+ = [0, +\infty). \tag{1.2}$$

Note that when $p(x) \equiv 1$, $w(x) \equiv 1$, the definition of (1.1) has been characterized by Savchuk and Shkalikov in [1] and [2]. When $w \equiv 1$, in [3] Goriunov and Mikhailets proposed the definition of (1.1) by introducing the quasi-derivative

$$f^{[1]}(x) = p(x)y'(x) - u(x)y(x).$$

By a similar method of the definition proposed by Goriunov and Mikhailets, we can define the singular Sturm–Liouville operators with distributional potential in the weighted Lebesgue spaces. Rewriting expression (1.1)

$$L_\upsilon y = \frac{1}{w}\left(-\big(y^{[1]}\big)' - \frac{u(x)}{p(x)}y^{[1]} - \frac{u^2(x)}{p(x)}y\right), \quad x \in \mathbb{R}_+, \tag{1.3}$$

we define the minimal operator

$$\mathbf{H}_\upsilon := \overline{\mathbf{H}^0_\upsilon},$$

where $\mathbf{H}^0_\upsilon y := L_\upsilon y$,

$$\operatorname{Dom}\big(\mathbf{H}^0_\upsilon\big) := \big\{f \in L^2_{w,\mathrm{comp}}(\mathbb{R}_+) : f,\ f^{[1]} \in W^{1,1}_{loc}(\mathbb{R}_+),\ f(0) = 0,\ L_\upsilon f \in L^2_w(\mathbb{R}_+)\big\}.$$

Here $L^2_{w,\mathrm{comp}}$ denotes the set of functions $\{f : \int_0^\infty w|f|^2 dx < +\infty$ and having compact support in $\mathbb{R}_+\}$ and $W^{n,p}_{loc}(\mathbb{R}_+) = \{f : f \in W^{n,p}[0, R]$ for all $R > 0\}$, where $W^{n,p}$ denotes the usual Sobolev space.

In the last decades, Schrödinger operators with distributional potentials have attracted tremendous interest since they can be used as solvable models in many situations. In particular, if υ is a linear combination of δ-functions, $\upsilon(x) = \sum_{k=1}^\infty \alpha_k \delta(x - x_k)$, the operator $\mathbf{H}_\upsilon$ describes δ-interactions of strength α_k at the points x_k (numerous results can be found in [4,5] and comprehensive lists of references therein). In this paper, we restrict our considerations to the case of operators with δ-type potentials, since such operators are known to yield physically interesting models.