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Canonical form and symmetry group of systems of conservation laws

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Relation between existence of an entropy and the symmetry group of a system of conservation laws is explored. For systems equipped with a single entropy, the familiar symmetric form of the system in divergence form is seen to be a useful framework for description of the symmetry group. Generalization of the symmetric form and divergence form is found appropriate for systems with multiple entropies, or for systems with a single entropy, but for which the dimension on the system exceeds that of the underlying phase space.

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1. Introduction

Multidimensional first-order systems of conservation or balance laws are widely adopted mathematical models. Here we address the combination or modification of such systems so as to obtain hyperbolicity and a desired symmetry group for the resulting system.

However desirable or necessary in a particular application, such features may prove elusive. A well-known example is the combination of copies of the Euler system, by choice of the equations of state, to obtain two-fluid or multi-fluid models [3]. Indeed, application to this problem, previously reported [13], largely motivated the present study. A second example is the combination of an Euler system with Maxwell's equations to obtain models of magnetohydrodynamic flow. As we shall discuss below, the Lundquist system [8,4,2], constructed on physical grounds, satisfies such criteria but nonetheless remains controversial.

For a first-order system of conservation or balance laws, the existence of a strictly convex entropy density implies hyperbolicity [5], and provides an admissibility condition and a crude bound for

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nonunique weak solutions of the related Cauchy problem [7]. Additionally, such systems necessarily admit a “symmetric” or “gradient” form [6,9].

Our point of departure is that the symmetric form of a given system, well-known to illuminate the existence of an entropy extension, has repeatedly been found useful, however ad hoc, in discussion of the related symmetry group [10–13]. Here we undertake a more systematic treatment of this phenomenon, combining applications to specific problems with more general results.

Using notation developed in Section 2, in Section 3 we extend the familiar symmetric form to systems with multiple entropies and to systems for which the dimension of the system exceeds that of the corresponding phase space. Systems with multiple entropies appear, for example, as those with a variational formulation, the multiple entropies resulting from application of Noether’s theorem. In contrast, below systems with a single entropy are those for which the entropy density/flux transforms like a single vector field under coordinate transformations. Systems for which the dimension of the symmetric form exceeds that of phase space arise when the entropy extension applies only to a subset of solutions. Such happens for example in the Lundquist system, for which the thermodynamic entropy relation applies only to solutions with the magnetic field initially (and of course for all time) solenoidal.

In Section 4, we establish how the symmetry properties of a given system, necessarily reflected in those of the entropy or entropies, are also necessarily reflected in those of the symmetric dependent variables and the related potential functions. As discussed in Section 5, the results simplify considerably in the case of systems with a single entropy.

As a first application of these results, we show in Section 6 how for a linear system, prescription of the symmetry group can severely restrict the form of the system.

In Section 7 we apply the results to the problem of finding a reduced or simplified form of a given system, with the same symmetry group and the same or a closely related entropy set. Familiar examples are the isentropic and incompressible Euler systems; a less familiar example is the construction of hyperbolic two-fluid models by an affine relation between the two reciprocal fluid temperatures [13]. These examples illustrate a more general principle established here, that for systems with a single entropy density/flux, an equation in the primitive system, corresponding to an invariant symmetric dependent variable, may be judiciously removed without loss of the entropy or the symmetry group.

The vector of potential functions associated with the symmetric form of both the relativistic and nonrelativistic Euler systems assumes a simple and convenient form. In Section 8 we show that this is not accidental, that at least within a suitable region of phase space, a Lorentz-rotation symmetric system with a single entropy and potential functions of this form is necessarily the relativistic Euler system.

Sections 9 and 10 are devoted to two classes of systems with multiple entropies. In each case, the symmetry group of the system is illuminated by generalization of the canonical form in which the system is expressed. In particular, the familiar “divergence form” of such systems is expeditiously replaced by expressions of closed differential forms of various orders. These results are used in Section 11 to characterize Galilean symmetric approximations of Maxwell’s equations.

As a final application, in Section 12 we recover the Lundquist system by combination of the Euler system with one of the Maxwell approximations. This treatment exposes controversial features of this system as a model of magnetohydrodynamic flow. Using results obtained here, alternative models of magnetohydrodynamic flow are constructed in a companion paper.

2. Notation and preliminaries

We discuss systems with “primitive form”

$$\sum_{i=0}^{d-1} a_{ji,x_i} = b_j, \quad j = 1, \ldots, n \tag{2.1}$$

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