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Regularity results on the parabolic Monge–Ampère equation with *VMO* type data

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ABSTRACT

This paper establishes interior estimates for L^p-norms, Orlicz norms of solutions to the parabolic Monge–Ampère equation $-u_t \det D^2 u = f(x,t)$ provided that $f(x,t)$ is positive, bounded, and satisfies a *VMO*-type condition, and some integrability conditions for f_t. Our results improve the corresponding results in Gutiérrez and Huang (2001) [8] in some sense.

1. Introduction

This paper is concerned with interior regularity of weak solutions to the parabolic Monge–Ampère equation

$$-u_t \det D^2 u = f(x,t) \quad \text{in } Q = \Omega \times (0,T], \tag{1.1}$$

where $u = u(x,t)$ is parabolically convex in Q, that is, convex in x and nonincreasing in t, $D^2 u$ denotes the Hessian of u with respect to x, Ω is a bounded convex domain in $\mathbb{R}^n$, and $f(x,t)$ is positive, bounded, and satisfies a *VMO*-type condition.

The regularity of solutions for the parabolic Monge–Ampère equation has been studied by many authors; see [6,7,10,14–16,18,19]. In particular, Gutiérrez and Huang [8] recently investigated interior

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regularity of solution to (1.1) in the case that f is continuous. For the elliptic Monge–Ampère equation, see [1–5,9,11].

Our purpose in this paper is to consider L^p-norms and Orlicz norms of solutions to (1.1), where $f(x,t)$ satisfies a *VMO*-type condition which may be discontinuous. For the elliptic Monge–Ampère equation, the estimates of this type were established in [12].

Before stating the main results in this paper, as in [12], we recall spaces $VMO(Q)$ and $BMO^{\Psi}(Q)$ (redenoted by $VMO^{\Psi}(Q)$ if $\Psi(0)=0$), and introduce local spaces $VMO_{loc}(Q,u)$ and $VMO^{\Psi}_{loc}(Q,u)$.

Let Ψ be a nondecreasing continuous function on $[0,\infty)$ such that $\Psi(t)>0$ for $t>0$ and $t/\Psi(t)$ is almost increasing, which means $t/\Psi(t)\leqslant Ks/\Psi(s)$ for $0<t<s$. For $f\in L^1(Q)$ and $A\subset Q$, the mean oscillation of $f(x,t)$ over A is defined by

$$\operatorname{mosc}_A f=\frac{1}{|A|}\int_A \left|f(x,t)-f_A\right|dx\,dt,$$

where f_A denotes the average of f over A. Let $z_0=(x_0,t_0)$ and $Q_{(r)}(z_0)=B_r(x_0)\times(t_0-r^2,t_0)$, where $B_r(x_0)$ is the ball centered at x_0 with radius r.

A function $f\in L^1(Q)$ belongs to $BMO_{\Psi}(Q)$ if there exists a constant C such that

$$\operatorname{mosc}_{Q_{(r)}(z_0)\cap Q} f\leqslant C\Psi(r)$$

for all $z_0\in Q$, $0<r\leqslant d=\operatorname{diam}(Q)$, where $\operatorname{diam}(Q)$ is the diameter of Q. We recall that $f\in VMO(Q)$ if and only if $\operatorname{mosc}_{Q_{(r)}(z_0)\cap Q} f$ converges to 0 uniformly in $z_0\in Q$ as $r\to 0$. For further properties of $BMO_{\Psi}(Q)$, see [11,12]. If $\Psi=1$, $BMO_{\Psi}(Q)$ is the usual $BMO(Q)$ space. Obviously, if $f\in BMO_{\Psi}(Q)$ with $\Psi(0)=0$, then f has vanishing mean oscillation of modulus $C\Psi$. For this reason, we set $VMO^{\Psi}=BMO_{\Psi}(Q)$ if $\Psi(0)=0$.

To introduce $VMO_{loc}(Q,u)$ and $VMO^{\Psi}_{loc}(Q,u)$, we recall the notation of sections of a parabolically convex function u.

For $z_0=(x_0,t_0)\in Q$, the section $Q_h(z_0)=Q_h(u,z_0)$ is defined by

$$Q_h(z_0)=\left\{(x,t)\in Q:\ u(x,t)\leqslant l_{z_0}(x)+h \text{ and } t\leqslant t_0\right\},$$

where $l_{z_0}(x)=u(x_0,t_0)+Du(x_0,t_0)(x-x_0)$.

Given a function $f\in L^1(Q)$, we say that $f\in VMO_{loc}(Q,u)$ if for any $Q'\Subset Q$

$$Q_f\left(r,Q'\right)=\sup_{z_0\in Q',\ \operatorname{diam} Q_h(u,z_0)\leqslant r}\operatorname{mosc}_{Q_h(u,z_0)} f\to 0,\quad \text{as } r\to 0.$$

A function $f\in L^{n+1}(Q)$ is said to be in $VMO^{\Psi}_{loc}(Q,u)$ where $\Psi(0)=0$ if for any $\Omega'\times(\epsilon,T]$ with $\Omega'\Subset\Omega$ and $0<\epsilon<T$, there exists $C=C(Q')$ such that for all $z_0\in Q'$ and $Q_h(u,z_0)\subset Q$

$$\operatorname{mosc}^{(n+1)}_{Q_h(u,z_0)} f\leqslant C\Psi\left(\operatorname{diam}\left(Q_h(u,z_0)\right)\right),$$

where $\operatorname{mosc}^{(n+1)}_A f=\left(\frac{1}{|A|}\int_A|f(x,t)-f_A|^{n+1}\,dx\,dt\right)^{1/(n+1)}$. Denote by $[f]_{VMO^{\Psi}(Q',u)}$ the smallest of all such constants $C(Q')$. The following theorem is the main results of this paper.

Theorem A. *Let $u=u(x,t)$ be parabolically convex and a solution to (1.1) in the cylinder $Q=\Omega\times(0,T]$ with $u=\varphi$ on $\partial_p Q=\partial\Omega\times(0,T]\cup\bar{\Omega}\times\{0\}$. Assume*

(A1) *$B_{n^{-3/2}}(0)\subset\Omega\subset B_1(0)$ convex, $\partial\Omega\in C^{1,\alpha}$ with $\alpha>1-2/n$; and $\exp(A(-f_t)^+)\in L^1(Q)$ for some $A>0$, and*