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Inverse problems: Dense nodal subset on an interior subinterval

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ABSTRACT

The inverse nodal problem for the Sturm–Liouville problems defined on interval $[0, 1]$ with separated boundary conditions is considered. We prove that a twin dense subset of the nodal set in interior subinterval $[a_1, a_2] (\subset [0, 1])$ uniquely determines the potential on $[0, 1]$ and the boundary conditions, through two cases of $1/2 \in [a_1, a_2]$ and $1/2 \notin [a_1, a_2]$. Note that, for the latter, we need additional spectral information, which is associated with the derivatives of eigenfunctions at some known nodal points.

1. Introduction

We are concerned with the Sturm–Liouville problems, denoted by $L := L(q; \alpha, \beta)$, consisting of the equation

$$-y'' + q(x)y = \lambda y \tag{1.1}$$

on $[0, 1]$ and boundary conditions

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0, \tag{1.2}$$

$$y(1)\cos\beta + y'(1)\sin\beta = 0. \tag{1.3}$$

Here the potential $q \in L^1[0, 1]$ is a real-valued function and $\alpha, \beta \in [0, \pi)$. It is well known [13] that the problem has a discrete spectrum consisting of simple real eigenvalues, denoted by $\sigma(L) = \{\lambda_n\}_{n=1}^{\infty}$.

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For each eigenvalue λ_n, the corresponding eigenfunction $y(x, \lambda_n)$ has exactly $(n-1)$ interior nodes (or zeros), denoted by $\{x_n^j\}_{j=1}^{n-1}$.

The inverse nodal problem, first posed and solved by McLaughlin [15,9], is the problem of finding the potential q and boundary conditions α, β using only the set of nodal points of eigenfunctions. In the past years, the inverse nodal problem of the Sturm–Liouville problems has been investigated by several authors (see [4,11,19,21,23] and the references therein). The investigation has also been extended to other problems, such as differential pencils and integro-differential operators, etc. (see [1,2,12] and the references therein). The known results contain the uniqueness, the reconstructing formula, the numerical scheme, and the stability. However, the uniqueness results show that the inverse nodal problem is overdetermined. In particular, Yang [22] proved an interesting theorem on inverse nodal problems which shows that the twin and s-dense subset of nodal points on the subinterval $[0, a_1]$ $(1/2 < a_1 \leqslant 1)$ is sufficient to determine $(q; \alpha, \beta)$ uniquely by using one of Gesztesy and Simon's results in [7]. Cheng, Law and Tsay [3] improved the theorem by replacing the condition of s-dense with dense. Indeed, Yang's theorem is based on the fact that the dense nodal subset is sufficient to uniquely recover the eigenvalues associated with the known nodal points. Moreover, for this transformed inverse spectral problem, the condition of $a_1 > 1/2$ is necessary by virtue of Gesztesy and Simon's counterexample (see [7, p. 2773]).

The aim of this paper is to provide a generalization of Yang's theorem that the known nodal subset only needs to be dense on an interior subinterval, $[a_1, a_2]$, of $[0, 1]$. More precisely, we prove that the unknown coefficients $(q; \alpha, \beta)$ are uniquely determined (up to a constant) by a twin dense subset of the nodal set on $[a_1, a_2]$ with $1/2 \in (a_1, a_2)$. In particular, the interval length $(a_2 - a_1)$ can be arbitrarily small.

We also consider the case of $1/2 \notin [a_1, a_2]$. Note that, in this case, the eigenvalues (of a spectrum) corresponding to the known nodes are not sufficient to recover the potential q uniquely on a more than half interval of $[0, 1]$, that is, $[0, a_1]$ or $[1 - a_2, 1]$. Therefore, we need additional spectral information to deal with this uniqueness problem. We will put forward $\{y'(x_{n_k}^{j_k}, \lambda_{n_k})\}_{k=1}^{\infty}$ as a new class of spectral data, where $y(x, \lambda)$ is a solution of Eq. (1.1) (see Section 2 below for details). This spectral data is analogous to the norming constants used in [17] and the interior spectral data in [16]. Here, in our opinion, we call it the interior spectral data corresponding to the Sturm–Liouville problem defined by (1.1)–(1.3). Together with this data, we prove that a twin dense subset of the nodal set on $[a_1, a_2]$ uniquely determines $(q; \alpha, \beta)$ (see Theorem 2.6 below). Thus, the open question raised by Yang in [22, p. 636] is solved effectively by means of this additional spectral information.

The strategy we use to prove our results is to convert the inverse nodal problem into the inverse spectral problem with partial information given on the potential on an interior subinterval. The converting is based on the asymptotic expressions of the eigenvalues and nodal points, and a relationship (see Lemma 3.2 below) between the eigenvalues and the Lebesgue points of potential functions, which can be regarded as a new approach to solve inverse nodal problems. Moreover, we employ common nodal information and the Riccati-type equations (cf., for example, [18]) associated with the Weyl m-functions to treat the inverse eigenvalue problem when partial information of the potential is known on an interior subinterval.

The organization of this paper is as follows. In Section 2, we provide the main results of this paper, which present a complete discussion for the nodal subset known on an interior subinterval. In Section 3, we recall some classical results and establish a relationship between the eigenvalues and the Lebesgue points of potentials. Section 4 contains the proofs of our uniqueness theorems.

2. Main results

In this section, we will give the main results of this paper. Let us mention that, hereafter, by the statement "the potential q on $[0, 1]$ and boundary data (α, β) are uniquely determined by the nodal subset A," we mean that there are no two different $L(q; \alpha, \beta)$ and $\tilde{L}(\tilde{q}; \tilde{\alpha}, \tilde{\beta})$ such that A is their common nodal subset.

Let λ_n be the nth eigenvalue of the problem (1.1)–(1.3). Let $0 < x_n^1 < \cdots < x_n^{n-1} < 1$ be the nodal points corresponding to their eigenfunction $y(x, \lambda_n)$, $l_n^j = x_n^{j+1} - x_n^j$ be the associated nodal length and