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Long time dynamics of highly concentrated solitary waves for the nonlinear Schrödinger equation

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Abstract

In this paper we study the behavior of solutions of a nonlinear Schrödinger equation in presence of an external potential, which is allowed to be singular at one point. We show that the solution behaves like a solitary wave for long time even if we start from an unstable solitary wave, and its dynamics coincides with that of a classical particle evolving according to a natural effective Hamiltonian.

Keywords: Solitary waves dynamics; Singular potential; Effective Hamiltonian

1. Introduction and statement of the results

In this paper we study the long time dynamics of a solitary wave solution of a nonlinear Schrödinger equation (NLS) in presence of an external potential. This problem has been considerably studied in the last years, following the tradition of the work on the stability of solitons which dates back to Weinstein [18].

The first dynamical results are given in [8] and improved, along the same lines, in [15]. This first approach is purely variational and is based on the non-degeneracy conditions proved in [17] for the ground state of the elliptic equation solved by the function describing the profile of a soliton. This approach has been used also in [6], where the results of [8,15] are extended to the case of a potential with a singularity.

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A second line of investigations on our problem has been initiated in [10,11]. In these papers the authors have strongly used the Hamiltonian nature of NLS, approximating the solution by its symplectic projection on the finite dimensional manifold of solitons (see (2.3), which is a submanifold of that used in [10,11], since we fix the profile U). This approach has been improved in [13,14] for the Gross–Pitaevskii equation by showing that it is possible to obtain an exact dynamics for the center of the soliton approximation.

In the previous papers the non-degeneracy condition for the ground state is a fundamental assumption. It has been removed in a more recent approach introduced in [3,4]. The idea of these papers is that it is possible for the solution of the NLS to remain concentrated for long time and to have a soliton behavior, even if the profile of the initial condition is degenerate for the energy associated to the elliptic equation. In fact the concentration of the solution follows in the semi-classical regime from the role played by the nonlinear term, which in [3,4] is assumed to be dependent on the Planck constant. This approach has been used in [7] for the NLS with a Hartree nonlinearity, in which case the non-degeneracy of the ground state is for the moment an open question.

In this paper we put together the last two approaches and try to weaken as much as possible the assumptions on the solitary wave. First of all, one main difference is that we control only the L^2 norm of the difference between the solution of NLS and the approximating traveling solitary wave. This has been done also in [1], and allows to drop the non-degeneracy condition and consider more general nonlinearities. Moreover we prove that the approximation of the solution of NLS with a traveling solitary wave is good also if the solitary wave is not stable, that is it is not a soliton, and the profile is fixed. This choice partly destroys the symplectic structure used in [10] and subsequent papers, but we prove that there exists a particular projection on the manifold $\mathcal{M}_\varepsilon$ defined in (2.3) which is almost symplectic for long time. Actually this particular projection is natural, since it is defined in terms of the Hamiltonian functional of NLS restricted to the manifold $\mathcal{M}_\varepsilon$, called the *effective Hamiltonian* in [14]. Then, this almost symplectic projection is enough to prove that the approximation is good for long time. Finally, we remark that we are able to consider the cases of regular and singular external potentials at the same time, and slightly improve on the range of allowed behavior at the singularity with respect to [6].

In the remaining part of this section we describe the problem and the main result and discuss the assumptions. In Section 2 we use the Hamiltonian nature of NLS to introduce the effective Hamiltonian on the manifold $\mathcal{M}_\varepsilon$ and to find the "natural" projection of the solution on $\mathcal{M}_\varepsilon$. In Sections 3 and 4 we describe the approximation of the solution of NLS and prove the main result. Finally in Appendix A we show that our projection is almost symplectic for long time.

1.1. The problem and the assumptions

We study the behavior of solutions $\psi(t,\cdot) \in H^1(\mathbb{R}^N,\mathbb{C})$, with $N \geq 3$ to the initial value problem

$$\begin{cases} i\varepsilon\psi_t + \varepsilon^2\triangle\psi - f\big(\varepsilon^{-2\alpha}|\psi|^2\big)\psi = V(x)\psi \\ \psi(0,x) = \varepsilon^\gamma U\big(\varepsilon^{-\beta}(x-a_0)\big)e^{\frac{i}{\varepsilon}(\frac{1}{2}(x-a_0)\cdot\xi_0+\theta_0)} \end{cases} \tag{$\mathcal{P}_\varepsilon$}$$

where $\varepsilon > 0$ represents the Planck constant, α, β, γ are real parameters, $(a_0, \xi_0) \in \mathbb{R}^N \times \mathbb{R}^N$ are the initial conditions of the finite dimensional dynamics which the solution follows, $\theta \in \mathbb{R}$ is the phase shift. Moreover $U \in H^1(\mathbb{R}^N)$ is a positive function which satisfies