Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 258 (2015) 906–918

Journal of Differential Equations

www.elsevier.com/locate/jde

Stability of symbolic embeddings for difference equations and their multidimensional perturbations

Hung-Ju Chen [a], Ming-Chia Li [b,*]

[a] Department of Economics, National Taiwan University, No. 1, Sec. 4, Roosevelt Road, Taipei 106, Taiwan
[b] Department of Applied Mathematics, National Chiao Tung University, 1001 University Road, Hsinchu 300, Taiwan

Received 12 September 2011; revised 28 September 2014

Available online 28 October 2014

Abstract

In this paper, we study complexity of solutions of a high-dimensional difference equation of the form

$$\Phi(x_{i-m}, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_{i+n}) = 0, \quad i \in \mathbb{Z},$$

where Φ is a C^1 function from $(\mathbb{R}^\ell)^{m+n+1}$ to $\mathbb{R}^\ell$. Our main result provides a sufficient condition for any sufficiently small C^1 perturbation of Φ to have symbolic embedding, that is, to possess a closed set of solutions Λ that is invariant under the shift map, such that the restriction of the shift map to Λ is topologically conjugate to a subshift of finite type. The sufficient condition can be easily verified when Φ depends on few variables, including the logistic and Hénon families. To prove the result, we establish a global version of the implicit function theorem for perturbed equations. The proof of the main result is based on the Brouwer fixed point theorem, and the proof of the global implicit function theorem is based on the contraction mapping principle and other ingredients. Our novel approach extends results in [2,3,8,15,21].

MSC: 39A33; 37B10; 37B45; 37J40; 26B10; 58C15

Keywords: Symbolic embedding; Multidimensional perturbation; Difference equation; Implicit function theorem

* Corresponding author.
E-mail addresses: hjc@ntu.edu.tw (H.-J. Chen), mcli@math.nctu.edu.tw (M.-C. Li).

http://dx.doi.org/10.1016/j.jde.2014.10.008
0022-0396/

1. Introduction

Consider a high-dimensional difference equation of the following form

$$\Phi(x_{i-m}, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_{i+n}) = 0, \quad i \in \mathbb{Z}. \tag{1}$$

where x_j's are variables in $\mathbb{R}^\ell$ and Φ is a function from $(\mathbb{R}^\ell)^{m+n+1}$ to $\mathbb{R}^\ell$. If Φ is reduced to a function φ depending on smaller number of variables, e.g., $\Phi = \varphi(x_i)$, we say that Φ has the *singular form* φ, and regards Φ and its perturbations as multidimensional perturbation of φ. In [2,8], the authors dealt with the case when $\ell = 1$ and Φ is the limit of a continuous one-parameter family Ψ_λ of functions as λ tends to λ_0. If Φ has a singular form φ of one variable in $\mathbb{R}$, and φ has $k \geq 2$ simple zeros, in [8] it was shown that among solutions of Ψ_λ with λ close to λ_0, there is a topological k-horseshoe, i.e., the two-sided full Bernoulli shift with k symbols; also refer to [3,21]. If Φ has a singular form of two variables, i.e., $\Phi = \varphi(x_i, x_{i+J})$, where $0 < |J| \leq \min(m, n)$, and for the equation $\varphi(x, y) = 0$ there is a branch $y = \xi(x)$ with positive topological entropy $h_{\mathtt{top}}(\xi)$, it was proved in [2] that for λ close to λ_0, there is a subset of solutions of Ψ_λ to which the restriction of the two-sided shift map has topological entropy arbitrarily close to $h_{\mathtt{top}}(\xi)/|J|$. Therein, the approach is based on hyperbolic region of interval maps, and is an extension of the anti-integrable limit approach introduced in [1] by dropping the requirement of Lagrangian generator. Unfortunately, there is an obstacle to extend such an approach to the case when $\ell > 1$ or Φ has a singular form of more than two variables. This is why we develop a novel approach in the present paper.

In this paper, we provide a sufficient condition for any sufficiently small C^1 perturbation of Φ to have *symbolic embedding*, that is, to possess a closed set of solutions Λ that is invariant under the two-sided full shift map, such that the restriction of the shift map to Λ is topologically conjugate to a subshift of finite type, associated to some transition matrix M. The sufficient condition requires existence of finitely many compact subsets K_j's of $\mathbb{R}^\ell$ such that any allowable product $\prod_{j\in\mathbb{Z}} K_{s_j}$ for M contains a solution $(x_j)_{j\in\mathbb{Z}}$ of (1) and satisfies $\det(\partial\Phi/\partial x_i) \neq 0$ on $\prod_{j=-m}^{n} K_{s_{i+j}}$; referred as '$\Phi$ has *nondegenerate coding* of M' in Definition 1.

The nondegenerate coding condition can be easily verified when Φ has a singular form of few variables. For instance, if, as above, Φ has a singular form φ of one variable in $\mathbb{R}$, and φ has $k \geq 2$ simple zeros, by simply taking K_j's to be small closed intervals containing zeros in the interiors, and M to be the $k \times k$ matrix with all entries equal to one, then Φ has nondegenerate coding of M and hence any sufficiently small C^1 perturbation of Φ has symbolic embedding of M; this extends results in [3,8,15,21]. The extension of results in [2] is presented at the end of next section. There are examples of singular forms of two or three variables, including the logistic and Hénon families, having nondegenerate coding of some transition matrix, and as an application of our result, the lower bound of topological entropy of high-dimensional Hénon-like maps is also obtained.

Our result justifies the use of low-dimensional models of complex phenomena. Consider the map

$$T(x_1, \cdots, x_m) = \big(F(x_1, \cdots, x_m), g_1(x_1), \cdots, g_{m-1}(x_{m-1})\big),$$

where the first component F can be reduced to a low-dimensional map f depending on the first few variables. Denote by $(x_{i,1}, \cdots, x_{i,m})$ the i-th iteration of an initial point $p = (x_{0,1}, \cdots, x_{0,m})$