Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 258 (2015) 980–1014

Journal of Differential Equations

www.elsevier.com/locate/jde

Optimal results on TV bounds for scalar conservation laws with discontinuous flux

Shyam Sundar Ghoshal [a,b,c,*]

[a] *Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, Paris, France*
[b] *Institut de Mathématiques de Toulouse, Université Paul Sabatier, Toulouse, France*
[c] *Gran Sasso Science Institute (GSSI), L'Aquila, Italy*

Received 13 May 2014

Available online 7 November 2014

Abstract

This paper is concerned with the total variation of the solution of scalar conservation law with discontinuous flux in one space dimension. One of the main unsettled questions concerning conservation law with discontinuous flux was the boundedness of the total variation of the solution near interface. In [1], it has been shown by a counter-example at $T = 1$, that the total variation of the solution blows up near interface, but in that example the solution become of bounded variation after time $T > 1$. So the natural question is what happens to the BV-ness of the solution for large time. Here we give a complete picture of the bounded variation of the solution for all time. For a uniform convex flux with only L^∞ data, we obtain a natural smoothing effect in BV for all time $t > T_0$. Also we give a counter-example (even for a BV data) to show that the assumptions which have been made are optimal.

Keywords: Hamilton–Jacobi equation; Scalar conservation laws; Discontinuous flux; Explicit formula; Characteristic lines; BV function

* Correspondence to: Gran Sasso Science Institute, viale Francesco Crispi, 7, 67100 L'Aquila (AQ), Italy.
E-mail addresses: s.s.ghoshal@gmail.com, ghoshal@ann.jussieu.fr, Shyam.Ghoshal@math.univ-toulouse.fr, shyam.ghoshal@gssi.infn.it.

http://dx.doi.org/10.1016/j.jde.2014.10.014
0022-0396/

1. Introduction

In this paper, we investigate the total variation bound of the following scalar conservation laws

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} F(x,u) = 0 \quad \text{if } x \in \mathbb{R},\ t > 0,$$
$$u(x,0) = u_0(x) \quad \text{if } x \in \mathbb{R}, \tag{1.1}$$

where the flux function $F(x,u)$ is a discontinuous function of x given by $F(x,u) = H(x)f(u) + (1 - H(x))g(u)$, H is the Heaviside function. Here we consider f, g to be strictly convex functions with superlinear growth. That is

$$\lim_{|u|\to\infty} \left(\frac{f(u), g(u)}{|u|} \right) = (\infty, \infty). \tag{1.2}$$

Eq. (1.1) has been extensively studied since few decades both from the theoretical and numerical points of view. Notice that a very few results are known regarding the total variation of the solution of (1.1) near interface.

A conservation law with a discontinuous flux of the form (1.1) is a first order hyperbolic model, which arises in many applicative problems. It has a huge application in fluid flows in heterogeneous media such as two-phase flow in a porous medium, which arises in the petroleum industry. Eq. (1.1) also arises while dealing with modeling gravity, continuous sedimentation in a clarifier–thickener unit [16,17,15,14,23,24]. Some other applications are in the model of car traffic flow on a highway (see [47]) and in ion etching in the semiconductor industry (see [48]).

It is well understood that even if F and u_0 are smooth, the solution of (1.1) may not admit classical solution in finite time, hence one must define the notion of weak solutions. In general, weak solutions of (1.1) are not unique. Due to this fact, one has to put some extra condition, so-called "entropy condition", in order to get the uniqueness. When $f = g$, Kružkov [40] has introduced the most general entropy condition in order to prove the uniqueness by using doubling variable technique. For $f \neq g$, the general entropy condition has been established in [5,8,7,18, 10,45] in order to prove the uniqueness.

When $f = g$, existence of the solution has been studied in several ways, namely vanishing viscosity method (see [40]), convergence of numerical schemes (see [30]), front tracking method (see [31]) and via Hamilton–Jacobi equation (see [25,42,49]).

In general, when $f \neq g$, (1.1) may not admit any solutions, hence for existence, some extra assumptions are required. Under the assumption that the fluxes f and g coincide at least two points Gimse–Risebro [28,27], Diehl [22] obtained a solution for Riemann data (also see [39, 23]). A theory by using front tracking method has been developed in [27,35,39]. Under the assumptions that the fluxes f and g are strictly convex and C^2, (1.1) also has been studied in [5]. The authors obtained a Lax–Oleinik type formula satisfying the following interface entropy condition

$$\text{meas}\{t:\ f'(u^+(t)) > 0,\ g'(u^-(t)) < 0\} = 0 \tag{1.3}$$

and Lax–Oleinik entropy condition for $x \neq 0$. Also they proves the L^1-contraction semi-group. Cauchy problem for the Hamilton–Jacobi equation with discontinuous coefficient has been settled by Ostrov [53]. On the other hand, Karlsen, Risebro, Towers [35,36] used a modified