Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 257 (2014) 1814–1839

Journal of Differential Equations

www.elsevier.com/locate/jde

On a nonclassical fractional boundary-value problem for the Laplace operator

Mykola Krasnoschok, Nataliya Vasylyeva *

Institute of Applied Mathematics and Mechanics of NAS of Ukraine, R. Luxemburg, str. 74, 83114 Donetsk, Ukraine

Received 29 August 2013; revised 6 May 2014

Available online 17 June 2014

Abstract

In this paper we consider a boundary-value problem for the Poisson equation with a boundary condition comprising the fractional derivative in time and the right-hand sides dependent on time. We prove the one-valued solvability of this problem, and provide the coercive estimates of the solution.

MSC: primary 35R11, 35C15; secondary 35B65, 35R35

Keywords: Elliptic equations; Fractional dynamic boundary condition; Caputo derivative; Coercive estimates

1. Introduction

Let a and b be some positive constants, $\nu \in (0, 1)$,

$$R^2_+ = \{(x_1, x_2) : x_1 \in (-\infty, +\infty), x_2 > 0\}, \qquad R^2_{+,T} = R^2_+ \times (0, T); \qquad R^1_T = R^1 \times (0, T).$$

In this paper we analyze the following problem with a fractional temporal derivative in a boundary condition:

* Corresponding author.

E-mail addresses: krasnoschok@iamm.ac.donetsk.ua (M. Krasnoschok), vasylyeva@iamm.ac.donetsk.ua (N. Vasylyeva).

http://dx.doi.org/10.1016/j.jde.2014.05.022

$$\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}=f_0(x_1,x_2,t) \quad \text{in } R^2_{+,T}; \tag{1.1}$$

$$u(x_1,0,t)=a\rho(x_1,t)+f_1(x_1,t) \quad \text{on } \bar{R}^1_T; \tag{1.2}$$

$$\mathbf{D}_t^\nu \rho(x_1,t)-b\frac{\partial u}{\partial x_2}=f_2(x_1,t) \quad \text{on } \bar{R}^1_T; \tag{1.3}$$

$$\rho(x_1,0)=0 \quad \text{in } R^1, \tag{1.4}$$

where f_i, $i=\overline{0,2}$, are some given functions, $\mathbf{D}_t^\nu$ denotes the Caputo fractional derivative with respect to t and is defined by (see, for example, (2.4.6) in [22])

$$\mathbf{D}_t^\nu w(\cdot,t)=\frac{1}{\Gamma(1-\nu)}\frac{\partial}{\partial t}\int_0^t \frac{w(\cdot,\tau)d\tau}{(t-\tau)^\nu}-\frac{w(\cdot,0)}{\Gamma(1-\nu)t^\nu}, \qquad \nu\in(0,1), \tag{1.5}$$

where $\Gamma(\nu)$ is the Gamma function.

Analysis of problem (1.1)–(1.4) is important in order to study the free boundary problem for the Laplace equation in the case of subdiffusion (the fractional quasistationary Stefan problem or the fractional Hele-Shaw problem [45]). We recall that the anomalous diffusion means that the diffusive motion cannot be modeled as standard Brownian motion [8,33], and the mean square displacement of the diffusing species $\langle(\Delta x)^2\rangle$ scales as a nonlinear power law in time, i.e. $\langle(\Delta x)^2\rangle \sim t^\nu$ for some real number ν. If $\nu\in(0,1)$, this is referred as a subdiffusion.

The fractional Hele-Shaw problem describes the evolution of fluid which is subjected to the "fractional" Darcy law [35], i.e. the "fractional" fluid velocity is proportional to the pressure gradient. This problem arises in controlled drug release system [30]; in studying of materials with memory [46]; in the transport processes at the Earth's surface [45]. The mathematical model of the fractional Hele-Shaw problem is the following [46].

Let $\Omega(t)$ be a bounded domain in R^2 for every $t\in[0,T]$, with the boundary $\partial\Omega=\Gamma\cup\Gamma(t)$, $\Gamma\cap\Gamma(t)=\emptyset$. Here Γ is a given fixed curve and $\Gamma(t)$ is an unknown boundary (free boundary). Assume that the equation of the free boundary is described as follows

$$\Phi(y,t)=0,$$

where $\Phi(y,t)$ is an unknown function.

We look for the fluid pressure $p(y,t)$, $y\in\Omega(t)$, $t\in[0,T]$, and the free boundary $\Gamma(t)$ by the following conditions:

$$\Delta_y p=0 \quad \text{in } \Omega(t); \qquad p=g(y,t) \quad \text{on } \Gamma; \tag{1.6}$$

$$p=0, \quad \text{and} \quad \mathbf{D}_t^\nu\Phi(y,t)=-\mu(\nabla_y p,\nabla_y\Phi) \quad \text{on } \Gamma(t),\ \nu\in(0,1); \tag{1.7}$$

$$\Omega(0) \quad \text{is given,} \tag{1.8}$$

where $g(y,t)$ is a given function, $\mathbf{D}_t^\nu\Phi(y,t)$ is the "fractional" velocity of the free boundary $\Gamma(t)$ in the direction of the outward normal to $\Omega(t)$, μ is some positive constant.

If $\nu=1$, problem (1.6)–(1.8) is interpreted as the mathematical model of the classical Hele-Shaw problem which has been intensively studied for more than half a century. The literature on this and related problems is vast. A bibliography containing references up to the late nineties has