Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 257 (2014) 1879–1938

Journal of Differential Equations

www.elsevier.com/locate/jde

Control problems for weakly coupled systems with memory

Paola Loreti, Daniela Sforza *

Dipartimento di Scienze di Base e Applicate per l'Ingegneria Sezione di Matematica, Sapienza Università di Roma, Via Antonio Scarpa 16, 00161 Roma, Italy

Received 14 August 2013; revised 5 February 2014

Available online 10 June 2014

Abstract

We investigate control problems for *wave–Petrovsky* coupled systems in the presence of memory terms. By writing the solutions as Fourier series, we are able to prove Ingham type estimates, and hence reachability results. Our findings have applications in viscoelasticity theory and linear acoustic theory.

MSC: 42A38; 45K05; 93C20

Keywords: Coupled systems; Convolution kernels; Fourier series; Ingham estimates; Reachability

1. Introduction

We will analyze control problems for *wave–Petrovsky* weakly coupled systems in the presence of memory terms. In particular, we will solve the reachability to a given target in a finite time, by using a harmonic approach based on Ingham type estimates.

In the papers [24,25] we studied reachability problems for a class of integro-differential equations

* Corresponding author.
E-mail addresses: paola.loreti@sbai.uniroma1.it (P. Loreti), daniela.sforza@sbai.uniroma1.it (D. Sforza).

http://dx.doi.org/10.1016/j.jde.2014.05.016

$$u_{tt}(t,x) - u_{xx}(t,x) + \beta \int_0^t e^{-\eta(t-s)} u_{xx}(s,x)\, ds = 0, \quad t \in (0,T),\ x \in (0,\pi),$$

then generalized to spherical domains in [26] and to more general kernels in [29].

The interest for researching this type of control problems comes from the theory of viscoelasticity. Exponential kernels naturally arise in linear viscoelasticity theory, such as in the analysis of Maxwell fluids or Poynting–Thomson solids, see e.g. [32,34]. Despite of our paper that can be fitted in the classical theory of controllability, it is worth to mention that another type of control problems regards the possibility that the stresses, in addition to the motion, can be controlled. This different statement is motivated by many applications and for this reason considered very significative, see [34].

For other references in viscoelasticity theory see the seminal papers of Dafermos [2,3] and [33,18]. For other type of kernels, see [30].

As it is well known, viscoelastic relaxation kernels have to be completely monotone functions, that is, continuously differentiable to every order functions $K(t)$ satisfying

$$(-1)^n K^{(n)}(t) \ge 0 \quad \forall n \in \mathbb{N},\ \forall t \ge 0.$$

This class of relaxation kernels includes, as a significant case, the Prony sum

$$K(t) = \sum_{i=1}^{N} \beta_i e^{-\eta_i t}$$

with $\beta_i > 0$ and $\eta_i \ge 0$, $i = 1, \dots, N$. Prony-sum kernels have many implications for the dispersion and the attenuation phenomena in acoustic theory [7,8,36]. Moreover, the analysis of the 1-d wave equation of a vibrating string has analogies with seismic wave propagation [37]. It could be interesting to consider in the model the effect of viscosity as an attenuation phenomenon for seismic events.

Continuing along the lines traced by the research papers [24–26], we have done further investigations, which split into the following three directions a), b) and c).

a) The study of a more general relaxation kernel of Prony type in a single wave equation. This problem presents some difficulties with respect to the case of kernels consisting in a single exponential function, because we have to handle a more complicated spectral analysis, to compare the coefficients of the materials and to find conditions under which the reachability control problem may have a positive solution ([27], in preparation).
b) The analysis of weakly coupled systems of *wave–wave* type, with a memory term having a single-exponential kernel as in [25]. To find the eigenvalues, one has to study a fifth-degree equation: it turns out that the two couples of complex conjugate roots have the same asymptotic behavior ([28], preprint). See [12] for one of the first papers on wave–wave coupled PDE's without memory.
c) The study of weakly coupled systems of *wave–Petrovsky* type, again with memory terms consisting in a single-exponential kernel. The analysis of weakly coupled PDE's of wave–Petrovsky type without memory began in [13], where the harmonic analysis approach was successfully applied to get observability results.