Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 257 (2014) 2102–2114

Journal of Differential Equations

www.elsevier.com/locate/jde

On the finiteness of uniform sinks

Dawei Yang [1], Yong Zhang [*,2]

School of Mathematical Sciences, Soochow University, Suzhou, 215006, PR China

Received 23 November 2013; revised 14 May 2014

Available online 5 June 2014

Abstract

We study the finiteness of uniform sinks for flows. Precisely, we prove that, for $\alpha > 0$ and $T > 0$, if a vector field X has only hyperbolic singularities or sectionally dissipative singularities, then X can have only finitely many (α, T)-uniform sinks. This is a generalized version of a theorem of Liao [4].

MSC: 37C10; 37C27; 37D05; 37D10; 37D30

1. Introduction

In this work, we give a generalized version of a theorem of Liao [4]. It could be seen as an extension of the remarkable Pliss' theorem [7] in the setting of singular flows. A natural question of dynamical systems is when we can have the coexistence of infinitely many periodic sinks. Newhouse [6] constructed an example for locally C^2 generic surface diffeomorphisms. On the other hand, under the star condition,[3] Pliss [7] proved that one can only have finitely many periodic sinks for diffeomorphisms or non-singular flows. For singular flows, this was known by Liao [4].

* Corresponding author.
E-mail addresses: yangdw1981@gmail.com, yangdw@suda.edu.cn (D. Yang), yongzhang@suda.edu.cn (Y. Zhang).
1 D. Yang was partially supported by NSFC 11001101, NSFC 11271152, Ministry of Education of PR China 20100061120098.
2 Y. Zhang was partially supported by NSFC 11271152.
3 This is a weaker condition than structural stability.

http://dx.doi.org/10.1016/j.jde.2014.05.028

Let M be a compact smooth Riemannian manifold and X be a smooth vector field on M. We know that X will generate a smooth flow ϕ_t. If $X(\sigma)=0$, σ is called a *singularity* of X. If $\phi_t(p)=p$ for some $t>0$ and $X(p)\neq 0$, p is called a *periodic point*. We use $\mathrm{Sing}(X)$ and $\mathrm{Per}(X)$ to denote the sets of singularities and periodic points.

The flow $\Phi_t=d\phi_t: TM\to TM$ is called the *tangent flow*. Note that every periodic orbit has at least one zero Lyapunov exponent w.r.t. Φ_t. To understand the dynamics in a small neighborhood of a periodic orbit, Poincaré used the Poincaré return map: for any point in the periodic orbit, one takes a cross section at that point, then the flow defines a local diffeomorphism in a small neighborhood of the cross section. The dynamics of the flow in a small neighborhood of the periodic orbit can be understood by the dynamics of the diffeomorphism.

By extending this idea to the general non-periodic case, for any regular point x and any $t\in\mathbb{R}$, one considers local normal cross sections at x and $\phi_t(x)$, then the flow gives a local diffeomorphism between these two cross sections. Its linearization is the *linear Poincaré flow* ψ_t, which is defined as the following: given a regular point $x\in M$, consider a vector v in the orthogonal complement of $X(x)$, one defines

$$\psi_t(v)=\Phi_t(v)-\frac{\langle\Phi_t(v),X(\phi_t(x))\rangle}{|X(\phi_t(x))|^2}X\big(\phi_t(x)\big).$$

Note that ψ_t cannot be defined on the singularities.

Given $\alpha>0$ and $T>0$, a periodic orbit Γ is called an *(α,T)-uniform sink* if there are $m\in\mathbb{N}$ and times $0=t_0<t_1<t_2\cdots<t_n=m\pi(\Gamma)$ ($\pi(\Gamma)$ is the period of Γ) satisfying $t_i-t_{i-1}\le T$ for any $1\le i\le n$ such that for any $x\in\Gamma$, one has

$$\prod_{i=1}^{n}\big\|\psi_{t_i-t_{i-1}}\big(\phi_{t_{i-1}}(x)\big)\big\|\le \mathrm{e}^{-\alpha m\pi(\Gamma)}.$$

A singularity σ is called *sectionally dissipative* if the following is true: when we list all the eigenvalues of $DX(\sigma)$ as $\{\lambda_1,\lambda_2,\cdots,\lambda_d\}$, we have $\mathrm{Re}(\lambda_i)+\mathrm{Re}(\lambda_j)\le 0$ for any $1\le i<j\le d$. Here d is the dimension of the manifold M.

Notice that for a sectionally dissipative non-hyperbolic singularity, the maximal of the real parts of its eigenvalues should be zero.

Theorem A. *Let $\alpha>0$, $T>0$. If a vector field X has only hyperbolic singularities or sectionally dissipative singularities, then X can have only finitely many (α,T)-uniform sinks.*

Similar results for diffeomorphisms or non-singular flows were got by Pliss [7]. If X has no singularities and X has infinitely many (α,T)-uniform sinks $\{\gamma_n\}$, then by using Pliss Lemma, for any γ_n, there is a point $x_n\in\gamma_n$ such that x_n has its stable manifold of uniform size which is independent with n. From the fact that M has finite volume, we can get a contradiction.

Liao [4] proved Theorem A with an additional assumption: X is a *star* vector field.[4] As X is star, every singularity of X is hyperbolic. If every singularity of X is hyperbolic and X has infinitely many (α,T)-uniform sinks γ_n, then by his estimation, for each n, there is a point

[4] For proving the finiteness of (α,T)-uniform sinks, Liao didn't use the full strength of the star condition: what he needed are the uniform estimations on sinks.