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J. Differential Equations 257 (2014) 2178–2224

Journal of Differential Equations

www.elsevier.com/locate/jde

Energy scattering for a Klein–Gordon equation with a cubic convolution

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Received 23 May 2012; revised 10 January 2014

Available online 10 June 2014

Abstract

In this paper, we study the global well-posedness and scattering problem in the energy space for both focusing and defocusing Klein–Gordon–Hartree equations in the spatial dimension $d \geqslant 3$. The main difficulties are the absence of an interaction Morawetz-type estimate and of a Lorentz invariance which enable one to control the momentum. To compensate, we utilize the strategy derived from concentration compactness ideas, which was first introduced by Kenig and Merle [13] to the scattering problem. Furthermore, employing technique from [32], we consider a virial-type identity in the direction orthogonal to the momentum vector so as to control the momentum in the defocusing case. While in the focusing case, we show that the scattering holds when the initial data (u_0, u_1) is radial, and the energy $E(u_0, u_1) < E(\mathcal{W}, 0)$ and $\|\nabla u_0\|_2^2 + \|u_0\|_2^2 < \|\nabla \mathcal{W}\|_2^2 + \|\mathcal{W}\|_2^2$, where $\mathcal{W}$ is the ground state.

MSC: primary 35P25; secondary 35B40, 35Q40, 81U99

Keywords: Klein–Gordon–Hartree equation; Scattering theory; Strichartz estimate

1. Introduction

This paper is devoted to the study of the Cauchy problem of the Klein–Gordon–Hartree equation

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http://dx.doi.org/10.1016/j.jde.2014.05.036
0022-0396/

$$\begin{cases} \ddot{u} - \Delta u + u + f(u) = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}^d,\ d \geqslant 3, \\ u(0,x) = u_0(x), & u_t(0,x) = u_1, \end{cases} \tag{1.1}$$

where $f(u) = \mu(|x|^{-\gamma} * |u|^2)u$, $2 < \gamma < \min(4, d)$, $\mu = \pm 1$ with $\mu = 1$ known as the defocusing case and $\mu = -1$ as the focusing case. Here u is a real-valued function defined in $\mathbb{R}^{d+1}$, the dot denotes the time derivative, Δ is the Laplacian in $\mathbb{R}^d$, $|x|^{-\gamma}$ is called the potential, and $*$ denotes the spatial convolution in $\mathbb{R}^d$.

Formally, the solution u of (1.1) conserves the energy

$$\begin{aligned} E\big(u(t), \dot{u}(t)\big) &= \frac{1}{2} \int_{\mathbb{R}^d} \big(\big|\dot{u}(t,x)\big|^2 + \big|\nabla u(t,x)\big|^2 + \big|u(t,x)\big|^2 \big) dx \\ &\quad + \frac{\mu}{4} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(t,x)|^2 |u(t,y)|^2}{|x-y|^\gamma} dx\, dy \\ &\equiv E(u_0, u_1), \end{aligned}$$

and the momentum

$$P(u)(t) = \int_{\mathbb{R}^d} u_t(t,x) \nabla u(t,x)\, dx = P(u)(0). \tag{1.2}$$

On the one hand, the scattering theory for the Klein–Gordon equation with $f(u) = \mu |u|^{p-1} u$ has been intensively studied in [3,4,7,10,29,31]. For $\mu = 1$ and

$$1 + \frac{4}{d} < p < 1 + \gamma_d \frac{4}{d-2}, \qquad \gamma_d = \begin{cases} 1, & 3 \leqslant d \leqslant 9; \\ \frac{d}{d+1}, & d \geqslant 10. \end{cases} \tag{1.3}$$

Brenner [4] established the scattering results in the energy space for $d \geqslant 10$. Thereafter, Ginibre and Velo [7] exploited the Birman–Solomjak space $\ell^m(L^q, I, B)$ in [2] and delicate estimates to improve the results in [4], which covered all subcritical cases. Finally K. Nakanishi [29] obtained the scattering results for the critical case by the strategy of induction on energy [6] and a new Morawetz-type estimate. And recently, S. Ibrahim, N. Masmoudi and K. Nakanishi [10,11] utilized the concentration compactness ideas to give the scattering threshold for the focusing nonlinear Klein–Gordon equation. Their method also works for the defocusing case.

On the other hand, the scattering theory for the Hartree equation

$$i\dot{u} = -\Delta u + \mu\big(|x|^{-\gamma} * |u|^2\big)u$$

has been also studied by many authors (see [5,9,16,19–24]). For the subcritical defocusing case, Ginibre and Velo [9] derived the associated Morawetz inequality and extracted a useful Birman–Solomjak type estimate to obtain the asymptotic completeness in the energy space. Nakanishi [30] improved the results by a new Morawetz estimate which doesn't depend on the nonlinearity. For the critical case, Miao, Xu and Zhao [19] took advantage of a new kind of the localized Morawetz estimate, which is also independent of the nonlinearity, to rule out the possibility of the energy concentration at origin and established the scattering results in the energy space for