



The “good” Boussinesq equation on the half-line

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Abstract

The initial–boundary value problem for the “good” Boussinesq (GB) equation on the half-line with data in Sobolev spaces is analysed via Fokas’ unified transform method and a contraction mapping approach. First, the basic space and time estimates for the linear GB initial value problem are derived and then the corresponding estimates for the initial–boundary value problem with zero initial data are obtained. Using these estimates, the Fokas solution formula for the linear GB on the half-line is shown to belong to appropriate Sobolev spaces. Finally, well-posedness of the nonlinear initial–boundary value problem is established by showing that the mapping defined by Fokas’ formula for GB, when the forcing is replaced with the Boussinesq nonlinearity, is a contraction mapping on a ball of the space $C([0, T^*]; H_x^s(0, \infty))$, $s > 1/2$, where the lifespan T^* depends on the size of the initial and boundary data. In addition, this work extends the validity of the solution formulas obtained by the unified method for the linear GB initial–boundary value problem to a broader Sobolev setting.

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1. Introduction

In 1872, the French mathematician Joseph Boussinesq derived approximations to the Euler equations under various regimes [1]. In the particular case of long, unidirectional waves (long in the sense of large wavelength compared to the water depth), Boussinesq was able to reduce his approximation to the single equation

$$u_{tt} - u_{xx} - u_{xxx} - (u^2)_{xx} = 0, \quad x \in \mathbb{R}, \quad t > 0. \quad (1.1)$$

This equation was an important discovery as it provided for the first time mathematical evidence in favour of John Scott Russell's observation of the solitary wave in 1834. Indeed, Eq. (1.1) admits the soliton travelling wave solution

$$u(x, t) = \frac{3}{2}(c^2 - 1)\operatorname{sech}^2\left[\frac{\sqrt{c^2 - 1}}{2}(x - ct)\right],$$

where $u(x, t)$ and c are the amplitude and the speed of the wave respectively.

Despite possessing travelling wave solutions of the type that Scott Russel had observed, Eq. (1.1) has a major disadvantage: it is ill-posed. This can be seen, for example, by seeking small amplitude solutions of the form

$$u(x, t) = \varepsilon e^{-ikx + i\omega t}, \quad \varepsilon \ll 1, \quad x \in \mathbb{R}, \quad t > 0, \quad k \in \mathbb{R}, \quad (1.2)$$

so that the nonlinear term of (1.1) is negligible. The dispersion relation in this case is given by $\omega^2 = k^2 - k^4$ and hence, for $|k| > 1$ the time frequency of (1.2) is imaginary and the wave amplitude grows exponentially at a rate of about $e^{k^2 t}$. In other words, the small initial datum $u_0(x) = \varepsilon e^{ikx}$ immediately evolves to the exponentially large solution $u(x, t)$ given by (1.2), justifying why Eq. (1.1) is nowadays known as the “bad” Boussinesq equation.

One way of resolving the issue of ill-posedness is to change the sign of the fourth derivative in (1.1) from negative to positive; the dispersion relation is now given by $\omega^2 = k^2 + k^4$, implying that $\omega \in \mathbb{R}$ for all $k \in \mathbb{R}$. The new equation is known as the “good” Boussinesq (GB) equation and reads

$$u_{tt} - u_{xx} + u_{xxx} + (u^2)_{xx} = 0, \quad x \in \mathbb{R}, \quad t > 0, \quad (1.3)$$

where we have also replaced u by $-u$ in order to get a positive sign in front of the nonlinear term, in agreement with the form usually appearing in the literature.

As the change of sign leading to the GB equation cannot be justified in the context of Boussinesq's physical derivation, Eq. (1.3) does not model nonlinear water waves. Nevertheless, it has been suggested by Zakharov [2] as a model of nonlinear vibrations along a string, and also by Turitsyn [3] for describing electromagnetic waves in nonlinear dielectric materials. Furthermore, Zakharov [2] also showed that both the “bad” and the “good” Boussinesq equations are integrable. In particular, by a simple change of variables they are both contained in the equation

$$u_{tt} - u_{xxx} + \lambda(u^2)_{xx} = 0, \quad \lambda = \pm 1, \quad (1.4)$$

which can be rewritten as the system

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