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Extremal functions for Trudinger–Moser inequalities of Adimurthi–Druet type in dimension two

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Abstract

Combining Carleson–Chang's result [9] with blow-up analysis, we prove existence of extremal functions for certain Trudinger–Moser inequalities in dimension two. This kind of inequality was originally proposed by Adimurthi and O. Druet [1], extended by the author to high dimensional case and Riemannian surface case [40,41], generalized by C. Tintarev to wider cases including singular form [36] and by M. de Souza and J.M. do Ó [14] to the whole Euclidean space $\mathbb{R}^2$. In addition to the Euclidean case, we also consider the Riemannian surface case. The results in the current paper complement that of L. Carleson and A. Chang [9], M. Struwe [35], M. Flucher [16], K. Lin [19], and Adimurthi and O. Druet [1], our previous ones [41,26], and part of C. Tintarev [36].

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1. Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^2$ and $W_0^{1,2}(\Omega)$ be the usual Sobolev space. The classical Trudinger–Moser inequality [44,33,32,37,30] says

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$$\sup_{u\in W_0^{1,2}(\Omega),\ \|\nabla u\|_2\le 1}\int_\Omega e^{4\pi u^2}dx<\infty. \tag{1}$$

Here and throughout this paper we denote the L^p-norm by $\|\cdot\|_p$. This inequality is sharp in the sense that for any $\alpha>4\pi$, the integrals in (1) are still finite but the supremum is infinite. Let $u_k\in W_0^{1,2}(\Omega)$ be such that $\|\nabla u_k\|_2=1$ and $u_k\rightharpoonup u$ weakly in $W_0^{1,2}(\Omega)$. Then P.L. Lions [20] proved that for any $p<1/(1-\|\nabla u\|_2^2)$, there holds

$$\limsup_{k\to\infty}\int_\Omega e^{4\pi p u_k^2}dx<\infty. \tag{2}$$

This inequality gives more information than the Trudinger–Moser inequality (1) in case $u\not\equiv 0$. While in case $u\equiv 0$, it is weaker than (1). However Adimurthi and O. Druet [1] proved that for any α, $0\le\alpha<\lambda_1(\Omega)$,

$$\sup_{u\in W_0^{1,2}(\Omega),\ \|\nabla u\|_2\le 1}\int_\Omega e^{4\pi u^2(1+\alpha\|u\|_2^2)}dx<\infty, \tag{3}$$

and that the supremum is infinity when $\alpha\ge\lambda_1(\Omega)$, where $\lambda_1(\Omega)$ is the first eigenvalue of the Laplacian operator with respect to Dirichlet boundary condition. For any sequence of functions $u_k\in W_0^{1,2}(\Omega)$ with $\|\nabla u_k\|_2=1$ and $u_k\rightharpoonup u$ weakly in $W_0^{1,2}(\Omega)$, if $u\not\equiv 0$, it then follows from (3) that for any α, $0\le\alpha<\lambda_1(\Omega)$,

$$\limsup_{k\to\infty}\int_\Omega e^{4\pi u_k^2(1+\alpha\|u_k\|_2^2)}dx<\infty. \tag{4}$$

Note that $1+\alpha\|u_k\|_2^2<1+\|\nabla u\|_2^2<1/(1-\|\nabla u\|_2^2)$ for sufficiently large k. (4) is weaker than (2). If $u\equiv 0$, we already see that (2) is weaker than (1), and obviously (4) is stronger than (1).

A natural question is to find the high dimensional analogue of (3). Let Ω be a smooth bounded domain in $\mathbb{R}^n$ $(n\ge 3)$. We proved in [40] that for any $0\le\alpha<\lambda_1(\Omega)$,

$$\sup_{u\in W_0^{1,n}(\Omega),\ \|\nabla u\|_n^n\le 1}\int_\Omega e^{\alpha_n|u|^{\frac{n}{n-1}}(1+\alpha\|u\|_n^n)^{\frac{1}{n-1}}}dx<\infty, \tag{5}$$

and that the supremum is infinite when $\alpha\ge\lambda_1(\Omega)$, where $\alpha_n=n\omega_{n-1}^{1/(n-1)}$, ω_{n-1} is the area of the unit sphere in $\mathbb{R}^n$, and $\lambda_1(\Omega)$ is defined by

$$\lambda_1(\Omega)=\inf_{u\in W_0^{1,n}(\Omega),\ u\not\equiv 0}\frac{\int_\Omega|\nabla u|^n dx}{\int_\Omega|u|^n dx}.$$

Trudinger–Moser inequalities on Riemannian manifolds were due to T. Aubin [7], J. Moser [30], P. Cherrier [12,13], and L. Fontana [17]. Also a few results was recently obtained, on complete noncompact Riemannian manifolds, by G. Mancini and K. Sandeep [27,28] and the