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Vector fields with homogeneous nonlinearities and many limit cycles

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Abstract

Consider planar real polynomial differential equations of the form $\dot{\mathbf{x}} = L\mathbf{x} + X_n(\mathbf{x})$, where $\mathbf{x} = (x, y) \in \mathbb{R}^2$, L is a 2×2 matrix and X_n is a homogeneous vector field of degree $n > 1$. Most known results about these equations, valid for infinitely many n, deal with the case where the origin is a focus or a node and give either non-existence of limit cycles or upper bounds of one or two limit cycles surrounding the origin. In this paper we improve some of these results and moreover we show that for $n \geq 3$ odd there are equations of this form having at least $(n + 1)/2$ limit cycles surrounding the origin. Our results include cases where the origin is a focus, a node, a saddle or a nilpotent singularity. We also discuss a mechanism for the bifurcation of limit cycles from infinity.
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1. Introduction and statement of the main results

For two dimensional real polynomial differential systems

$$\frac{dx}{dt} = \dot{x} = P(x, y), \qquad \frac{dy}{dt} = \dot{y} = Q(x, y), \quad (x, y) \in \mathbb{R}^2, \tag{1}$$

with $P(x, y), Q(x, y) \in \mathbb{R}[x, y]$ the ring of polynomials, the integer $n = \max\{\deg P, \deg Q\}$ is called the *degree* of the system. A *limit cycle* of system (1) is an isolated periodic solution in the set of all its periodic solutions. The second part of Hilbert's 16th problem [16] consists in determining a uniform upper bound on the number of limit cycles of all polynomial differential systems of degree n, together with the distribution of these maximum number of limit cycles. For more details see e.g. [9,17,15,23] and the references therein. As we know, this problem is still open even for $n = 2$.

Here we restrict our study to the existence and number of limit cycles surrounding the origin for the real planar polynomial differential systems with homogeneous nonlinearities and a singularity at the origin, i.e. of the form,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = L \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} P_n(x, y) \\ Q_n(x, y) \end{pmatrix}, \quad \text{where } L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{2}$$

$a, b, c, d \in \mathbb{R}$, and $P_n(x, y)$ and $Q_n(x, y)$ are homogeneous polynomials of degree $n \geq 2$. One of the particularities of this family is that each limit cycle that surrounds the origin can be expressed in polar coordinates as $r = R(\theta)$, for some smooth 2π-periodic function, see for instance [3–5, 8]. This particularity makes natural to face this very special and simpler case of Hilbert's 16th problem.

The number of limit cycles of (2) has been studied by many authors. When the origin is a focus, there are plenty of results, see for instance [3–6,10,12–14,18,21] and the references therein. But there are relatively few results when the origin is a node, a saddle, or a nilpotent singularity.

In [2] the authors studied system (2) with $b = c = 0$ and $a = d \neq 0$, and proved that if n is even the system has no limit cycles surrounding the origin, and that if n is odd then the system has at most one limit cycle, and there are examples of such systems which do have one limit cycle. We notice that this case is the simplest one, because in polar coordinates it writes as a Bernoulli equation and so it is integrable.

Consider now system (2) with $b = c = 0$ and $a \neq d$, $ad > 0$. Notice that $\lambda = a$ and $\mu = d$ are the eigenvalues of L and system (2) is written as

$$\dot{x} = \lambda x + P_n(x, y), \qquad \dot{y} = \mu y + Q_n(x, y), \tag{3}$$

with $\lambda\mu > 0$. In [5,19] it is proved that if n is even system (3) has no limit cycles. For n odd both papers provided some sufficient (and different) conditions under which the system has either no limit cycles or at most two limit cycles surrounding the origin. Examples of systems (3) having exactly either two, or one, or no limit cycles surrounding the origin already appear in Proposition 6.3 and Remark 6.4 of [11]. In this situation, results on the existence of at most two limit cycles are also given in [3, Thm. A].

In [8] the authors studied conditions for the existence of limit cycles (none, one, two or three) of differential systems defined by the sum of two quasi-homogeneous polynomial vector fields.

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