Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 257 (2014) 4369–4402

Journal of Differential Equations

www.elsevier.com/locate/jde

Delta shock waves with Dirac delta function in both components for systems of conservation laws ☆

Hanchun Yang *, Yanyan Zhang

Department of Mathematics, Yunnan University, Kunming 650091, PR China

Received 1 June 2014; revised 11 July 2014

Available online 10 September 2014

Abstract

We study a class of non-strictly and weakly hyperbolic systems of conservation laws which contain the equations of geometrical optics as a prototype. The Riemann problems are constructively solved. The Riemann solutions include two kinds of interesting structures. One involves a cavitation where both state variables tend to zero forming a singularity, the other is a delta shock wave in which both state variables contain Dirac delta function simultaneously. The generalized Rankine–Hugoniot relation and entropy condition are proposed to solve the delta shock wave. Moreover, with the limiting viscosity approach, we show all of the existence, uniqueness and stability of solution involving the delta shock wave. The generalized Rankine–Hugoniot relation is also confirmed. Then our theory is successfully applied to two typical systems including the geometric optics equations. Finally, we present the numerical results coinciding with the theoretical analysis.

Keywords: Hyperbolic systems of conservation laws; Geometric optics; Cavitation; Delta shock wave; Generalized Rankine–Hugoniot relation; Limiting viscosity approach

---

1. Introduction

This work is the twin of a recent Ref. [8]. In [8], we studied the Riemann problems of the hyperbolic systems of conservation laws

---

☆ Supported by the NSF of China 11361073.

* Corresponding author.

http://dx.doi.org/10.1016/j.jde.2014.08.009

$$\begin{cases} u_t + \big(\phi(u,v)u\big)_x = 0, \\ v_t + \big(\phi(u,v)v\big)_x = 0 \end{cases} \tag{1.1}$$

with Riemann initial data

$$(u,v)(0,x) = (u_\pm, v_\pm) \quad (\pm x > 0), \tag{1.2}$$

where $\phi(u,v)$ satisfies the assumption:

(H$_1$) $\phi(u,v) = \phi(au+bv)$ is a given smooth function satisfying $a^2+b^2 \neq 0$, a and b are constants.

A theory of delta shock wave with Dirac delta function developing in both state variables was established and applied to a large variety of systems. With regard to the delta shock wave and its related topics, we refer to [8] for a more detailed review. The theory of delta shock wave with Dirac delta function developing in both state variables is quite different from the aforementioned those, on which only one state variable contains the Dirac delta function. However, the theory does not work in some typical systems. For instance, the following equations of geometrical optics

$$\begin{cases} u_t + \left(\dfrac{u^2}{\sqrt{u^2+v^2}}\right)_x = 0, \\ v_t + \left(\dfrac{uv}{\sqrt{u^2+v^2}}\right)_x = 0 \end{cases} \tag{1.3}$$

were proposed by Engquist and Runborg [1] in 1996 when they considered multiple scale problems in numerical computations of propagating waves in vacuum medium, where (u,v) represents the direction and strength (particle density) of ray. The system (1.3) is only weakly hyperbolic. Generally, it means that (1.3) is not well-posed in the strongly hyperbolic sense. Besides, the system (1.3) possesses extreme degeneracy and strong nonlinearity in flux functions. These features may lead to a tough problem in the system (1.3). According to rough analysis and numerical evidence, they suggested that (1.3) can have delta function type solutions which may be used to describe the phenomenon where the phase should have split into two new phases. Since then, the existence of solutions to (1.3) has been an open question. The Riemann problem has not been solved so far. To our knowledge, some efforts have been done to explore the theoretical issues, for example [2,7] in which the authors tried to weaken the nonlinearity of flux functions by transformation of variables, but the results are far from the desired aim because the delta shock wave of (1.3) may contain Dirac delta function simultaneously in both state variables u and v. Fortunately, the problem has been fulfilled as below.

Solving Riemann problem (1.3), (1.2) is interesting and exciting, but it is just one of the objectives of the present paper. To this end, we focus here on establishing a theory of the delta shock wave for a class of systems (1.1) with the assumption:

(H$_2$) $\phi(u,v)$ is a smooth function satisfying $\phi(u,v) = \phi(\alpha u, \alpha v)$, $\alpha > 0$ is constant.

Obviously, in this situation, we can rewrite $\phi(u,v) = \phi(\frac{u}{v}) = \phi(r)$ $(r = \frac{u}{v})$ if $v \neq 0$. By taking $\phi(u,v) = \frac{u}{\sqrt{u^2+v^2}}$, one can find that the system (1.3) is the very prototype of (1.1).